# STRONG DIFFERENTIAL SUPERORDINATION

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Abstract. The notion of differential superordination was introduced in [5] by S.S. Miller and P.T. Mocanu as a dual concept of differential subordination [4]. The notion of strong differential subordination was introduced by J.A. Antonino, S. Romaguera in [1]. In this paper we introduce and study the concept of strong differential superordination following the general theory of differential superordinations presented in [5]. Let $\Omega$ be any set in the complex plane $\mathbb{C}$, let $p$ be analytic in the unit disk $U$ and let $\psi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. In this article we consider the problem of determining properties of functions $p$ that satisfy the strong differential superordination

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z, \xi) \mid z \in U,\ \xi \in \overline{U}\}.$$

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## 1. Introduction

Let $\Omega$ be any set in the complex plane $\mathbb{C}$, let $p$ be analytic in the unit disk $U$ and let $\psi(r, s, t; z, \xi) : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. In a series of articles, such as [6], the authors have determined properties of functions $p$ that satisfy the strong differential subordination

$$\{\psi(p(z), zp'(z), z^2p''(z); z, \xi) \mid z \in U,\ \xi \in \overline{U}\} \subset \Omega$$

In this article we consider the dual problem of determining properties of functions $p$ that satisfy the strong differential superordination

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z, \xi) \mid z \in U,\ \xi \in \overline{U}\}.$$

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of functions analytic in $U$. For $n$ a positive integer and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \{f \in \mathcal{H};\ f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \dots,\ z \in U\}.$$

Most of the functions considered in this article, and conditions on them are defined uniformly in the unit disk. Because of this we shall often omit the requirement $z \in U$ in most of the definitions and results. We shall indicate in those other cases when different domains are involved. For $0 < r < 1$, we let $U_r = \{z;\ |z| < r\}$.

## 2. Main results

Since many of the results in this article can be expressed in terms of strong subordination and strong superordination, we review here those definitions.

**Definition 1.** *Let $f$ be member of $\mathcal{H}$ and $F$ be analytic in $U \times \overline{U}$. The function $f$ is said to be subordinate to $F$, or $F$ is said to be superordinate to $f$, if there exists a function $w$ analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$ and such that $f(z) = F(w(z), \xi)$. In such a case we write $f \prec\prec F$ or $f(z) \prec\prec F(z, \xi)$. If $F(z, \xi)$ is univalent in $U$ for all $\xi \in \overline{U}$ then $f \prec\prec F$ if $f(0) = F(0, \xi)$, $\xi \in \overline{U}$ and $f(U) \subset F(U \times \overline{U})$.*

Let $\Omega$ and $\Delta$ be any sets in $\mathbb{C}$, let $p$ be analytic in the unit disk $U$ and let $\varphi(r, s, t; z, \xi) : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$. In this article we consider conditions on $\Omega$, $\Delta$ and $\varphi$ for which the following implication holds:

$$(1) \qquad \Omega \subset \{\varphi(p(z), zp'(z), z^2p''(z); z, \xi \mid z \in U,\ \xi \in \overline{U}\} \Rightarrow \Delta \subset p(U).$$

There are three distinct cases to consider in analyzing this implication, which we list as the following problems.

**Problem 1.** Given $\Omega$ and $\Delta$, find conditions on the function $\varphi$ so that (1) holds. We call such a $\varphi$ an **admissible function**.

**Problem 2.** Given $\varphi$ and $\Omega$, find a set $\Delta$ such that (1) holds. Furthermore, find the "largest" such $\Delta$.

**Problem 3.** Given $\varphi$ and $\Delta$, find a set $\Omega$ such that (1) holds. Furthermore, find the "smallest" such $\Omega$.

If either $\Omega$ or $\Delta$ in (1) is a simply connected domain, then (1) can be rephrased in terms of strong differential subordination.

If $p$ is analytic in $U$, and if $\Delta$ is a simply connected domain with $\Delta \neq \mathbb{C}$, then there is a conformal mapping $q$ of $U$ onto $\Delta$ such that $q(0) = p(0)$. In this case (1) can be rewritten as

$$(2) \qquad \Omega \subset \{\varphi(p(z), zp'(z), z^2p''(z); z, \xi \mid z \in U,\ \xi \in \overline{U}\} \Rightarrow q(z) \prec p(z).$$

If $\Omega$ is also a simply connected domain with $\Omega \neq \mathbb{C}$, then there is a conformal mapping $h$ of $U$ onto $\Omega$ such that $h(0) = \varphi(p(0), 0, 0; 0, \xi)$. If in addition, the function $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ is analytic in $U$, then (2) can be rewritten as

$$(3) \qquad h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi) \Rightarrow q(z) \prec p(z).$$

This implication also has meaning if $h$ and $q$ are analytic and not necessarily univalent. This last result leads us to some of the important definitions that will be used in this article.

**Definition 2.** *Let* $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ *and let* $h$ *be analytic in* $U$. *If* $p$ *and* $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ *are univalent in* $U$ *for all* $\xi \in \overline{U}$ *and satisfy the (second-order)* **strong differential superordination**

$$h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi) \tag{4}$$

*then* $p$ *is called a* **solution** *of the strong differential superordination.*

The analytic function $q$ is called a **subordinant of the solution of the strong differential superordination**, or more simply a **subordinant**, if $q \prec p$ for all $p$ satisfying (4).

A univalent dominant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants $q$ of (4) is said to be the best subordinant.

Note that the best subordinant is unique up to a rotation of $U$.

For $\Omega$ a set in $\mathbb{C}$, with $\varphi$ and $p$ as given in Definition 2, suppose (4) is replaced by

$$\Omega \subset \{\varphi(p(z), zp'(z), z^2p''(z); z, \xi) \mid z \in U,\ \xi \in \overline{U}\}. \tag{4'}$$

Although this more general situation is a **differential containment**, the condition in (4') will also be referred to as a **strong differential superordination**, and the definitions of solution, subordinant and best subordinant as given above can be extended to this generalization.

In the special case when the set inclusion (1) can be replaced by the strong superordination (3) we can reinterpret the three problems referred to above as follows:

**Problem 1'.** Given univalent functions $h$ and $q$ find a class of admissible functions $\phi[h, q]$ such that (3) holds.

**Problem 2'.** Given the strong differential superordination in (4) find a subordinant $q$. Moreover, find the best subordinant.

**Problem 3'.** Given $\varphi$ and subordinant $q$, find the largest class of univalent functions $h$ such that (3) holds.

Before obtaining our main result we need to introduce a class of univalent functions defined on the unit disk that have some nice boundary properties.

**Definition 3.**[4, Definition 2.2b, p.21] *We denote by* $Q$ *the set of functions* $q$ *that are analytic and injective on* $\overline{U} - E(f)$, *where*

$$E(f) = \left\{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\right\},$$

*and are such that* $f'(\zeta) \neq 0$ *for* $\zeta \in \partial U \setminus E(f)$.

The subclass of $Q$ for which $f(0) = a$ is denoted by $Q(a)$.

We will use the following Lemma [5, Lemma A] from the theory of differential superordinations to determine subordinants of strong differential superordinations.

**Lemma A.** [5].*Let* $p \in Q(a)$, *and let*

$$q(z) = a + a_n z^n + \dots$$

*be analytic in* $U$ *with* $p(z) \not\equiv a$ *and* $n \geq 1$. *If* $q$ *is not subordinate to* $p$, *then there exist points* $z_0 = r_0 e^{i\theta_0} \in U$ *and* $\zeta_0 \in \partial U \setminus E(p)$, *and an* $m \geq n \geq 1$ *for which* $q(U_{r_0}) \subset p(U)$,

*(i)* $q(z_0) = p(\zeta_0)$

*(ii)* $z_0 q'(z_0) = m\zeta_0 p'(\zeta_0)$ *and*

*(iii)* $\mathrm{Re}\dfrac{z_0 q''(z_0)}{q'(z_0)} + 1 \geq m\mathrm{Re}\left[\dfrac{\zeta_0 p''(\zeta_0)}{p'(\zeta_0)} + 1\right]$.

## 3. Admissible function and a fundamental result

We next define the class of admissible functions referred to in the introduction.

**Definition 4.** *Let* $\Omega$ *be a set in* $\mathbb{C}$, $q \in \mathcal{H}[a, n]$ *with* $q'(z) \neq 0$. *The* **class of admissible functions** $\Phi_n[\Omega, q]$, *consists of those functions* $\varphi : \mathbb{C}^3 \times U \times \overline{U} \to \mathbb{C}$ *that satisfy the admissibility condition:*

$$\varphi(r, s, t; \zeta, \xi) \in \Omega \qquad (A)$$

*whenever* $r = q(z)$, $s = \dfrac{zq'(z)}{m}$,

$$\mathrm{Re}\left[\frac{t}{s} + 1\right] \leq \frac{1}{m}\mathrm{Re}\left[\frac{zq''(z)}{q'(z)} + 1\right],$$

*where* $z \in U$, $z \in \partial U$, $\zeta \in \overline{U}$ *and* $m \geq n \geq 1$. *When* $n = 1$ *we write* $\phi_1[\Omega, q]$ *as* $\phi[\Omega, q]$.

In the special case when $h$ is an analytic mapping of $U$ onto $\Omega \neq \mathbb{C}$, we denote this class $\Phi_n[h(U), q]$ by $\Phi_n[h, q]$.

If $\varphi : \mathbb{C}^2 \times U \times \overline{U} \to \mathbb{C}$, then the admissibility condition $(A)$ reduces to

$$\varphi\left(q(z), \frac{zq'(z)}{m}; \zeta, \xi\right) \in \Omega \qquad (A')$$

where $z \in U$, $\zeta \in \partial U \setminus E(q)$, $\xi \in \overline{U}$ and $m \geq n \geq 1$.

The next theorem is a key result in the theory of first and second order strong differential superordinations. The proof is very short because of the use of Lemma

A and the very special conditions given in the definition of the class of admissible functions $\Phi_n[\Omega, q]$.

**Theorem 1.** *Let $\Omega \subset \mathbb{C}$, let $q \in \mathcal{H}[a,n]$ and let $\varphi \in \Phi_n[\Omega, q]$. If $p \in Q(a)$ and $\varphi(p(z), zp'(z), z^2p''(z); z, \xi)$ is univalent in $U$, for all $\xi \in \overline{U}$, then*

$$\Omega \subset \{\varphi(p(z), zp'(z), z^2p''(z); z, \xi) \mid z \in U,\ \xi \in \overline{U}\} \tag{5}$$

*implies*

$$q(z) \prec p(z), \quad z \in U.$$

*Proof.* Assume $q \not\prec p$. By Lemma A there exist points $z_0 = r_0e^{i\theta_0} \in U$, and $\zeta_0 \in \partial U \setminus E(q)$, and an $m \geq n \geq 1$ that satisfy (i)-(iii) of Lemma A. Using these conditions with $r = p(\zeta_0)$, $s = \zeta_0 p'(\zeta_0)$, $t = \zeta_0^2 p''(\zeta_0)$ and $\zeta = \zeta_0$ in Definition 5 we obtain

$$\varphi(p(\zeta_0), \zeta_0 p'(\zeta_0), \zeta_0^2 p''(\zeta_0); \zeta_0, \xi) \in \Omega. \tag{6}$$

Since $\zeta_0$ is a boundary point we deduce that (6) contradicts (5) and we must have $q(z) \prec p(z)$, $z \in U$. □

We next consider the special situation when $h$ is analytic on $U$ and $h(U) = \Omega \neq \mathbb{C}$. In this case, the class $\phi_n[h(U), q]$ is written as $\phi_n[h, q]$ and the following result is an immediate consequence of Theorem 1.

**Theorem 2.** *Let $q \in \mathcal{H}[a,n]$, let $h$ be analytic in $U$ and let $\varphi \in \phi_n[h, q]$. If $p \in Q(a)$ and $\varphi(p(z), zp'(z), z^2p''(z), z, \xi)$ is univalent in $U$, for all $\xi$, then*

$$h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z, \xi)$$

*implies*

$$q(z) \prec p(z), \quad z \in U.$$

**Example 1.** Let $h(z) = q(z) = \dfrac{1 + (2\alpha - 1)z}{1 + z}$, where $0 < \alpha < 1$ and

$$\mathrm{Re}[zp'(z) + p(z) + B(\xi)] > 0, \text{ for } 0 < \mathrm{Re}B(\xi) \leq \frac{1}{2}$$

If $p \in Q(1)$ then

$$\frac{1 + (2\alpha - 1)z}{1 + z} \prec\prec zp'(z) + p(z) + B(\xi), \quad z \in U,\ \xi \in \overline{U}$$

implies

$$\frac{1 + (2\alpha - 1)z}{1 + z} \prec p(z), \quad z \in U.$$

*Proof.* Since $h(U) = \{w \mid \text{Re}w > \alpha,\ 0 < \alpha < 1\} = \Omega$ and $\Omega \subset \{w \mid \text{Re}w > 0\}$ we have

$$\frac{1 + (2\alpha - 1)z}{1 + z} \prec p(z), \quad z \in U$$

from Theorem 2.

**Remark 1.**In [6, Example 1] we deduced: If $p \in \mathcal{H}[1, 1]$ and $0 < \text{Re}B(\xi) \le \frac{1}{2}$, then

$$zp'(z) + p(z) + B(\xi) \prec\prec \frac{1 + z}{1 - z}, \quad z \in U,\ \xi \in \overline{U}$$

implies

$$p(z) \prec \frac{1 + z}{1 - z}, \quad z \in U.$$

Using the conditions from Example 1 and Example 1 from [6], we obtain the following sandwich-type result:

If $p \in \mathcal{H}[1, 1]$ and $0 < \text{Re}B(\xi) \le \frac{1}{2}$, then

$$\frac{1 + (2\alpha - 1)^2}{1 + z} \prec\prec zp'(z) + p(z) + B(\xi) \prec\prec \frac{1 + z}{1 - z},$$

implies $\frac{1+(2\alpha-1)^2}{1+z} \prec p(z) \prec \frac{1+z}{1-z}, \quad z \in U.$

## References

[1] José A. Antonino and Salvador Romaguera, *Strong differential subordination to Briot-Bouquet differential equations*, Journal of Differential Equations 114(1994), 101-105.

[2] José A. Antonino, *Strong differential subordination and applications to univalency conditions*, J. Korean Math. Soc., **43**(2006), no.2, 311-322.

[3] S.S. Miller and P.T. Mocanu, *Differential subordinations. Theory and applications*, Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 2000.

[4] S.S. Miller and P.T. Mocanu, *Subordinants of differential superordinations*, Complex Variables, vol.48, no.10, 815-826.

[5] Georgia Irina Oros and Gh. Oros, *Strong differential subordination*, Turkish Journal of Mathematics, 32(2008), 1-11.

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