NEIGHBORHOOD AND PARTIAL SUM PROPERTY FOR UNIVALENT HOLOMORPHIC FUNCTIONS IN TERMS OF KOMATU OPERATOR

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Abstract. In the present paper, we investigate some important properties of a new class of univalent holomorphic functions by using Komatu operator. For example coefficient estimates, extreme points, neighborhoods and partial sums.

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1. Introduction and preliminaries

Let $\mathcal{S}$ denote the class of functions f that are analytic in the open unit disk $\mathcal{D} = \{z \in \mathcal{C} : \ |z| < 1\}$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

and consider the subclass $\mathcal{T}$ consisting of functions f which are univalent in $\mathcal{D}$ and are of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k. \tag{2}$$

For $\alpha \geq 0, \ 0 \leq \beta < 1, \ c \geq -1$ and $\delta \geq 0$, we let $\mathcal{S}_c^{\delta}(\alpha, \beta)$ be the subclass of $\mathcal{S}$ consisting of functions of the form (1) and satisfying the condition

$$Re \left\{ \frac{\mathcal{K}_c^{\delta}(f)}{z[\mathcal{K}_c^{\delta}(f)]'} \right\} > \alpha \left| \frac{\mathcal{K}_c^{\delta}(f)}{z[\mathcal{K}_c^{\delta}(f)]'} - 1 \right| + \beta. \tag{3}$$

The operator $\mathcal{K}_c^{\delta}(f)$ is the Komato operator [3] defined by

$$\mathcal{K}_c^{\delta}(f) = \int_0^1 \frac{(c+1)^{\delta}}{\Gamma(\delta)} t^c \left(\log \frac{1}{t} \right)^{\delta - 1} \frac{f(tz)}{t} dt.$$

We also let

$$\mathcal{TS}_c^{\delta}(\alpha,\beta)=\mathcal{S}_c^{\delta}(\alpha,\beta)\cap\mathcal{T}. \tag{4}$$

By applying a simple calculation for $f\in\mathcal{S}_c^{\delta}(\alpha,\beta)$ we get

$$\mathcal{K}_c^{\delta}(f)=z-\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\delta}|a_k|z^k. \tag{5}$$

The family $\mathcal{TS}_c^{\delta}(\alpha,\beta)$ is of special interest for contains many well-known as well as new classes of analytic functions. For example $\mathcal{TS}_c^{0}(0,\beta)$ is the family of starlike functions of order at most $\frac{1}{\beta}$.

In our present investigation, we need the following elementary Lemmas.

Lemma 1.1. *If $\alpha\geq 0,\ 0\leq\beta<1$ and $\gamma\in\mathcal{R}$ then Re $\omega>\alpha|\omega-1|+\beta$ if and only if Re $[\omega(1+\alpha e^{i\gamma})-\alpha e^{i\gamma}]>\beta$ where ω be any complex number.*

Lemma 1.2. *With the same condition in Lemma 1.1, Re $\omega>\alpha$ if and only if $|\omega-(1+\alpha)|<|\omega+(1-\alpha)|$.*

The main aim of this paper is to verify coefficient bounds and extreme points of the general class $\mathcal{TS}_c^{\delta}(\alpha,\beta)$. Furthermore, we obtain neighborhood property for functions in $\mathcal{TS}_c^{\delta}(\alpha,\beta)$. Also partial sums of functions in the class $\mathcal{S}_c^{\delta}(\alpha,\beta)$ are obtained.

2. Coefficient bounds for $\mathcal{S}_c^{\delta}(\alpha,\beta)$

In this section we find a necessary and sufficient condition and extreme points for functions in the class $\mathcal{TS}_c^{\delta}(\alpha,\beta)$.

Theorem 2.1. *If*

$$\sum_{k=2}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{(1-\beta)}\left(\frac{c+1}{c+k}\right)^{\delta}a_k<1, \tag{6}$$

then $f(z)\in\mathcal{S}_c^{\delta}(\alpha,\beta)$.

Proof. Let (6) hold, we will show that (3) is satisfied and so $f(z)\in\mathcal{S}_c^{\delta}(\alpha,\beta)$. By Lemma 1.2, it is enough to show that

$$|\omega-(1+\alpha|\omega-1|+\beta)|<|\omega+(1-\alpha|\omega-1|-\beta)|,$$

where $\omega=\frac{\mathcal{K}_c^{\delta}(f)}{z[\mathcal{K}_c^{\delta}(f)]'}$. By letting $B=\frac{z[\mathcal{K}_c^{\delta}(f)]'}{|z[\mathcal{K}_c^{\delta}(f)]'|}$ and by using (5) we may write

$$\begin{aligned} R &= |\omega+1-\beta-\alpha|\omega-1|| \\ &= \frac{1}{|z[\mathcal{K}_c^{\delta}(f)]'|}\left|2z-\beta z-\sum_{k=2}^{\infty}[1+(1-\beta)k+\alpha-\alpha a_k]\left(\frac{c+1}{c+k}\right)^{\delta}a_kz^k\right|. \end{aligned}$$

This implies that

$$R > \frac{|z|}{|z[\mathcal{K}_c^\delta(f)]'|}\left[2-\beta-\sum_{k=2}^{\infty}[k+(1+\alpha)-k(\alpha+\beta)]\left(\frac{c+1}{c+k}\right)^\delta a_k\right].$$

Similarly, if $L = |\omega - 1 - \alpha|\omega - 1| - \beta|$ we get

$$L < \frac{|z|}{|z[\mathcal{K}_c^\delta(f)]'|}\left[\beta+\sum_{k=2}^{\infty}[-k+(1+\alpha)-k(\alpha+\beta)]\left(\frac{c+1}{c+k}\right)^\delta a_k\right].$$

It is easy to verify that $R - L > 0$ if (6) holds and so the proof is complete.

Theorem 2.2. *Let $f \in \mathcal{T}$. Then f is in $\mathcal{TK}_c^\delta(\alpha,\beta)$ if and only if*

$$\sum_{k=2}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}\left(\frac{c+1}{c+k}\right)^\delta a_k < 1.$$

Proof. Since $\mathcal{T}$ is the subclass of $\mathcal{S}$ and $\mathcal{TS}_c^\delta(\alpha,\beta) = \mathcal{S}_c^\delta(\alpha,\beta)\cap\mathcal{T}$, and using Theorem 2.1, we need only to prove the necessity of theorem. Suppose that $f \in \mathcal{TK}_c^\delta(\alpha,\beta)$. By Lemma 1.1, and letting $\omega = \frac{\mathcal{K}_c^\delta(f)}{z[\mathcal{K}_c^\delta(f)]'}$ in (3) we obtain

$$Re\ (\omega(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma}) > \beta$$

or

$$Re\left[\frac{z-\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^\delta a_k z^k}{z\left(1-\sum_{k=2}^{\infty}k\left(\frac{c+1}{c+k}\right)^\delta a_k z^{k-1}\right)}(1+\alpha e^{i\gamma})-\alpha e^{i\gamma}-\beta\right] > 0,$$

then

$$Re\left[\frac{1-\beta-\sum_{k=2}^{\infty}(1-\beta k)\left(\frac{c+1}{c+k}\right)^\delta a_k z^k-\alpha e^{i\gamma}\sum_{k=2}^{\infty}(1-k)\left(\frac{c+1}{c+k}\right)^\delta a_k z^{k-1}}{\left(1-\sum_{k=2}^{\infty}k\left(\frac{c+1}{c+k}\right)^\delta a_k z^{k-1}\right)}\right] > 0.$$

The above inequality must hold for all z in $\mathcal{D}$. Letting $z = re^{-i\theta}$ where $0 \le r < 1$ we obtain

$$Re\left[\frac{1-\beta-\sum_{k=2}^{\infty}(1-\beta k)\left(\frac{c+1}{c+k}\right)^\delta a_k r^k-\alpha e^{i\gamma}\sum_{k=2}^{\infty}(1-k)\left(\frac{c+1}{c+k}\right)^\delta a_k r^{k-1}}{\left(1-\sum_{k=2}^{\infty}k\left(\frac{c+1}{c+k}\right)^\delta a_k r^{k-1}}\right)}\right] > 0.$$

By letting $r \to 1$ through half line $z = re^{-i\theta}$ $(0 \leq r < 1)$ and the mean value theorem we have

$$Re\ \left[(1-\beta) - \sum_{k=2}^{\infty}[(1-\beta k) - \alpha(1-k)]\left(\frac{c+1}{c+k}\right)^{\delta} a_k] > 0\right].$$

Therefore

$$\sum_{k=2}^{\infty}[(1-\beta k) + \alpha(1-k)]\left(\frac{c+1}{c+k}\right)^{\delta} a_k < 1-\beta.$$

This implies that

$$\sum_{k=2}^{\infty}\frac{(1+\alpha) - k(\alpha+\beta)}{(1-\beta)}\left(\frac{c+1}{c+k}\right)^{\delta} a_k < 1,$$

and the proof is complete.

Theorem 2.3. *Let* $f_1(z) = z$ *and*

$$f_k(z) = z - \frac{1-\beta}{[(1+\alpha) - k(\alpha+\beta)]}\left(\frac{c+k}{c+1}\right)^{\delta} z^k, \quad k \geq 2. \tag{7}$$

Then $f \in \mathcal{TS}_c^{\delta}(\alpha,\beta)$ *if and only if it can be expressed in the form* $f(z) = \sum_{k=1}^{\infty}\eta_k f_k(z)$ *where* $\eta_k \geq 0$ *and* $\sum_{k=1}^{\infty}\eta_k = 1$. *In particular, the extreme points of* $\mathcal{TS}_c^{\delta}(\alpha,\beta)$ *are the functions defined by (7).*

Proof. Let f be expressed as in the above theorem. This means that we can write

$$\begin{aligned} f(z) \quad &= \sum_{k=1}^{\infty}\eta_k f_k(z) = \eta_1 f_1(z) + \sum_{k=2}^{\infty}\eta_k f_k(z) \\ &= \eta_1 z + \sum_{k=2}^{\infty}\eta_k z - \sum_{k=2}^{\infty}\frac{(1-\beta)}{[(1+\alpha)-k(\alpha+\beta)]}\left(\frac{c+k}{c+1}\right)^{\delta}\eta_k z^k \\ &= z\sum_{k=1}^{\infty}\eta_k - \sum_{k=2}^{\infty}t_k z^k, \end{aligned}$$

where $t_k = \frac{(1-\beta)}{[(1+\alpha)-k(\alpha+\beta)]}\left(\frac{c+k}{c+1}\right)^{\delta}\eta_k$. Therefore $f \in \mathcal{TS}_c^{\delta}(\alpha,\beta)$ since

$$\sum_{k=2}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}t_k\left(\frac{c+1}{c+k}\right)^{\delta} = \sum_{k=2}^{\infty}\eta_k = 1-\eta_1 < 1.$$

Conversely, suppose that $f \in \mathcal{TS}_c^{\delta}(\alpha,\beta)$. Then by (6), we have

$$a_k < \frac{1-\beta}{[(1+\alpha)-k(\alpha+\beta)]}\left(\frac{c+k}{c+1}\right)^{\delta}, \quad k \geq 2.$$

So

$$\begin{aligned} f(z) &= z - \sum_{k=2}^{\infty} \frac{1-\beta}{[(1+\alpha)-k(\alpha+\beta)]} \left(\frac{c+k}{c+1}\right)^{\delta} \eta_k z^k \\ &= z - \sum_{k=2}^{\infty} \eta_k (z - f_k(z)) \\ &= \left(1 - \sum_{k=2}^{\infty} \eta_k\right) z - \sum_{k=2}^{\infty} \eta_k f_k(z) \\ &= \eta_1 z - \sum_{k=2}^{\infty} \eta_k f_k(z) = \sum_{k=1}^{\infty} \eta_k f_k(z). \end{aligned}$$

This completes the proof.

3.Neighborhood Property

In this section we study neighborhood property for functions in the class $\mathcal{TS}_c^{\delta}(\alpha,\beta)$. This concept was introduced by Goodman [2] and Ruscheweyh [6]. See also [1], [4], [5], and [7].

Definition 3.1. *For functions f belong f to $\mathcal{S}$ of the form (1) and $\gamma \geq 0$, we define $\eta-\gamma$-neighborhood of f by*

$$\mathcal{N}_{\gamma}^{\eta}(f) = \{g(z) \in \mathcal{S} : g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad \sum_{k=2}^{\infty} k^{\eta+1} |a_k - b_k| \leq \gamma\},$$

where η is a fixed positive integer.

By using the following lemmas we will investigate the $\eta-\gamma$-neighborhood of functions in $\mathcal{TS}_c^{\delta}(\alpha,\beta)$.

Lemma 3.2. *Let $m \geq 0$ and $-1 \leq \theta < 1$. If $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ satisfies*

$$\sum_{k=2}^{\infty} k^{\rho+1} |b_k| \leq \frac{1-\theta}{1+\alpha},$$

then $g(z) \in \mathcal{S}_c^{\rho}(\alpha,\theta)$.

Proof. By using of Theorem 2.1, it is sufficient to show that

$$\frac{(1+\alpha)-k(\alpha+\theta)}{1-\theta} \left(\frac{\rho+1}{\rho+k}\right)^{\delta} = \frac{k^{\rho+1}}{1-\theta}(1+\alpha).$$

But

$$\frac{(1+\alpha)-k(\alpha+\theta)}{1-\theta} \left(\frac{\rho+1}{\rho+k}\right)^{\delta} \leq \frac{1+\alpha}{1-\theta} \left(\frac{\rho+1}{\rho+k}\right)^{\delta}.$$

Therefore it is enough to prove that

$$Q(k,\rho)=\frac{\left(\frac{\rho+1}{\rho+k}\right)^{\delta}}{k^{\rho+1}}\leq 1.$$

The result follows because the last inequality holds for all $k\geq 2$.

Lemma 3.3. *Let* $f(z)=z-\sum_{k=2}^{\infty}a_kz^k\in\mathcal{T}$, $-1\leq\beta<1$, $\alpha\geq 0$ *and* $\epsilon\geq 0$. *If* $\frac{f(z)+\epsilon z}{1+\epsilon}\in\mathcal{TS}_c^{\delta}(\alpha,\beta)$ *then*

$$\sum_{k=2}^{\infty}k^{\rho+1}a_k\leq\frac{2^{\rho+1}(1-\beta)(1+\epsilon)}{(1-\alpha-2\beta)}\left(\frac{c+2}{c+1}\right)^{\delta}$$

where either $\rho=0$ *and* $c\geq o$ *or* $\rho=1$ *and* $1\leq c\leq 2$. *The result is sharp with the extremal function*

$$f(z)=z-\frac{(1-\beta)(1+\epsilon)}{(1-\alpha-2\beta)}\left(\frac{c+2}{c+1}\right)^{\delta}z^2,\quad(z\in\mathcal{D}).$$

Proof. Letting $g(z)=\frac{f(z)+\epsilon z}{1+\epsilon}$ we have

$$g(z)=z-\sum_{k=2}^{\infty}\frac{a_k}{1+\epsilon}z^k,\quad(z\in\mathcal{D}).$$

In view of theorem 2.3, $g(z)=\sum_{k=1}^{\infty}\eta_kg_k(z)$ where $\eta_k\geq 0$, $\sum_{k=1}^{\infty}\eta_k=1$,

$$g_1(z)=z\ \ \textit{and}\ \ g_k(z)=z-\frac{(1-\beta)(1+\epsilon)}{[(1+\alpha)-k(\alpha+\beta)]}\left(\frac{c+k}{c+1}\right)^{\delta}z^k\quad(k\geq 2).$$

So we obtain

$$\begin{aligned}g(z)\quad&=\eta_1z+\sum_{k=2}^{\infty}\eta_k\left[z-\frac{(1-\beta)(1+\epsilon)}{[(1+\alpha)-k(\alpha+\beta)]}\left(\frac{c+k}{c+1}\right)^{\delta}z^k\right]\\&=z-\sum_{k=2}^{\infty}\eta_k\left(\frac{(1-\beta)(1+\epsilon)}{[(1+\alpha)-k(\alpha+\beta)]}\left(\frac{c+k}{c+1}\right)^{\delta}\right)z^k.\end{aligned}$$

Since $\eta_k\geq 0$ and $\sum_{k=2}^{\infty}\eta_k\leq 1$, it follows that

$$\sum_{k=2}^{\infty}k^{\rho+1}a_k\leq\sup_{k\geq 2}k^{\rho+1}\left[\frac{(1-\beta)(1+\epsilon)}{[(1+\alpha)-k(\alpha+\beta)]}\left(\frac{c+k}{c+1}\right)^{\delta}\right].$$

Since whenever $\rho=0$ and $c\geq 0$ or $\rho=1$ and $1\leq c\leq 2$ we conclude

$$W(k,\rho,\alpha,\beta,\epsilon,c,\delta)=k^{\rho+1}\left[\frac{(1-\beta)(1+\epsilon)}{[(1+\alpha)-k(\alpha+\beta)]}\left(\frac{c+k}{c+1}\right)^{\delta}\right],$$

is a decreasing function of k, the result will follow. So the proof is complete.

Theorem 3.4. *Let either $\rho = 0$ and $c \geq 0$ or $\rho = 1$ and $1 \leq c \leq 2$. Suppose $-1 \leq \beta < 1$, and*

$$-1 \leq \theta < \frac{(1-\alpha-2\beta)(c+1)^{\delta} - 2^{\rho+1}(1-\beta)(1+\epsilon)(c+2)^{\delta}(1+\alpha)}{(1-\alpha-2\beta)(c+1)^{\delta}(1+\alpha)},$$

$f(z) \in \mathcal{T}$ and $\frac{f(z)+\epsilon z}{1+\epsilon} \in \mathcal{TS}_c^{\delta}(\alpha,\beta)$. Then the $\eta-\gamma$-neighborhood of f is the subset of $\mathcal{S}_c^{\eta}(\alpha,\theta)$, where

$$\gamma = \frac{(1-\theta)(1-\alpha-2\beta)(c+1)^{\delta} - 2^{\eta+1}(1-\beta)(1+\epsilon)(c+2)^{\delta}(1+\alpha)}{(1-\alpha-2\beta)(c+1)^{\delta}(1+\alpha)}.$$

The result is sharp.

Proof. For $f(z) = z - \sum_{k=2}^{\infty}|a_k|z^k$, let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ be in $\mathcal{N}_{\gamma}^{\eta}(f)$. So by Lemma 3.3, we have

$$\begin{aligned}\sum_{k=2}^{\infty} k^{\eta+1}|b_k| &= \sum_{k=2}^{\infty} k^{\eta+1}|a_k - b_k - a_k| \\ &\leq \gamma + \frac{2^{\eta+1}(1-\beta)(1+\epsilon)}{(1-\alpha-2\beta)}\left(\frac{c+2}{c+1}\right)^{\delta}.\end{aligned}$$

By using Lemma 3.2, $g(z) \in \mathcal{S}_c^{\eta}(\alpha,\beta)$ if

$$\gamma + \frac{2^{\eta+1}(1-\beta)(1+\epsilon)}{(1-\alpha-2\beta)}\left(\frac{c+2}{c+1}\right)^{\delta} \leq \frac{1-\theta}{1+\alpha}.$$

That is,

$$\gamma \leq \frac{(1-\theta)(1-\alpha-2\beta)(c+1)^{\delta} - 2^{\eta+1}(1-\beta)(1+\epsilon)(c+2)^{\delta}(1+\alpha)}{(1-\alpha-2\beta)(c+1)^{\delta}(1+\alpha)}$$

and the proof is complete.

4. Partial Sums

In this section we verify some properties of partial sums of functions in the class $\mathcal{S}_c^{\delta}(\alpha,\beta)$. (see [5] and [8])

Theorem 4.1. *Let $f(z) \in \mathcal{S}_c^{\delta}(\alpha,\beta)$, and define the partial sums $f_1(z)$ and $f_n(z)$ by*

$$f_1(z) = z \quad \textit{and} \quad f_n(z) = z + \sum_{k=2}^{n} a_k z^k \qquad (n \in \mathcal{N}, \quad n > 1). \tag{8}$$

If

$$\sum_{k=2}^{\infty} c_k|a_k| \leq 1 \tag{9}$$

where

$$c_k = \frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}\left(\frac{c+1}{c+k}\right)^{\delta}. \tag{10}$$

Then $f_k(z) \in \mathcal{S}_c^{\delta}(\alpha,\beta)$. *Moreover*

$$Re\left\{\frac{f(z)}{f_n(z)}\right\} > 1-\frac{1}{c_{n+1}}, \quad (z\in\mathcal{D},\ n\in\mathcal{N}) \tag{11}$$

$$Re\left\{\frac{f_n(z)}{f(z)}\right\} > \frac{c_{n+1}}{1+c_{n+1}}. \tag{12}$$

Proof. It is easy to show that $f_1(z) = z \in \mathcal{S}_c^{\delta}(\alpha,\beta)$. So by Theorem 3.3, and condition (9), we have $\mathcal{N}_1^{\eta}(z) \subset \mathcal{S}_c^{\delta}(\alpha,\beta)$, so $f_k \in \mathcal{S}_c^{\delta}(\alpha,\beta)$. Next, for the coefficient c_k it is easy to show that

$$c_{k+1} > c_k > 1.$$

Therefore by using (9) we obtain

$$\sum_{k=2}^{n}|a_k| + c_{n+1}\sum_{k=n+1}^{\infty}|a_k| \leq \sum_{k=2}^{\infty} c_k|a_k| \leq 1. \tag{13}$$

By putting

$$\begin{aligned}
h_1(z) \quad &= c_{n+1}\left\{\frac{f(z)}{f_n(z)} - \left(1-\frac{1}{c_{n+1}}\right)\right\}\\
&= 1 + c_{n+1}\left(\frac{f(z)}{f_n(z)}-1\right)\\
&= 1 + c_{n+1}\left(\frac{z+\sum_{k=2}^{\infty}a_kz^k}{z+\sum_{k=2}^{n}a_kz^k}-1\right) = 1 + c_{n+1}\left(\frac{1+\sum_{k=2}^{\infty}a_kz^{k-1}}{1+\sum_{k=2}^{n}a_kz^{k-1}}-1\right)\\
&= 1 + c_{n+1}\left[\frac{1+\sum_{k=2}^{\infty}a_kz^{k-1}-1-\sum_{k=2}^{n}a_kz^{k-1}}{1+\sum_{k=2}^{\infty}a_kz^{k-1}}\right]\\
&= 1 + \frac{c_{n+1}\sum_{k=n+1}^{\infty}a_kz^{k-1}}{1+\sum_{k=2}^{n}a_kz^{k-1}},
\end{aligned}$$

and using (13), for all $z \in \mathcal{D}$ we have

$$\left|\frac{h_1(z)-1}{h_1(z)+1}\right| = \left|\frac{\frac{c_{n+1}\sum_{k=n+1}^{\infty}a_kz^{k-1}}{1+\sum_{k=2}^{n}a_kz^{k-1}}}{2+\frac{c_{n+1}\sum_{k=n+1}^{\infty}a_kz^{k-1}}{1+\sum_{k=2}^{n}a_kz^{k-1}}}\right| \leq \frac{c_{n+1}\sum_{k=n+1}^{\infty}|a_k|}{2-2\sum_{k=2}^{n}|a_k| - c_{n+1}\sum_{k=n+1}^{\infty}|a_k|} \leq 1,$$

which proves (11). Similarly, if we put

$$\begin{aligned} h_2(z) &= \left\{ \frac{f_n(z)}{f(z)} - \frac{c_{n+1}}{1+c_{n+1}} \right\} (1+c_{n+1}) \\ &= 1 - \frac{(1+c_{n+1}) \sum_{k=n+1}^{\infty} a_k z^{k-1}}{1+\sum_{k=2}^{\infty} a_k z^{k-1}}, \end{aligned}$$

and using (13) we obtain

$$\left| \frac{h_2(z)-1}{h_2(z)+1} \right| \leq 1, \quad (z \in \mathcal{D}),$$

which yields the condition (12). So the proof is complete.

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