HOMOMORPHISMS BETWEEN C^*-ALGEBRAS AND THEIR STABILITIES

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Abstract. In this paper, we introduce the following additive type functional equation

$$f(rx+sy) = \frac{r+s}{2}f(x+y) + \frac{r-s}{2}f(x-y),$$

where $r, s \in \mathbb{R}$ with $r+s, r-s \neq 0$. Also we investigate the Hyers–Ulam–Rassias stability of this functional equation in Banach modules over a unital C^*-algebra. These results are applied to investigate homomorphisms between C^*-algebras.

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1. Introduction

A classical question in the theory of functional equations is the following: *When is it true that a function, which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?* If the problem accepts a solution, we say that the equation $\mathcal{E}$ is stable. Such a problem was formulated by Ulam [32] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [9]. It gave rise the stability theory for functional equations. Aoki [2] generalized the Hyers theorem for approximately additive mappings. Th.M. Rassias [28] extended the Hyers theorem by obtaining a unique linear mapping under certain continuity assumption when the Cauchy difference is allowed to be unbounded. P. Găvruta [7] provided a further generalization of the Th.M. Rassias theorem. For the history and various aspects of this theory we refer the reader to [26, 27, 29, 30]. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3], [4], [5], [8], [11] and [15]-[25]). We also refer the readers to the books [1], [6], [10], [12] and [31].

In this paper, we introduce the following additive functional equation

$$f(rx+sy) = \frac{r+s}{2}f(x+y) + \frac{r-s}{2}f(x-y), \tag{1}$$

where $r, s \in \mathbb{R}$ with $r+s, r-s \neq 0$. We investigate the Hyers–Ulam–Rassias stability of the functional equation (1) in Banach modules over a unital C^*-algebra. These results are applied to investigate homomorphisms between unital C^*-algebras.

2. Hyers–Ulam–Rassias stability of the functional equation (1) in Banach modules over a C^*-algebra

Throughout this section, assume that A is a unital C^*-algebra with norm $|.|$, unit 1. Also we assume that X and Y are (unit linked) normed left A-module and Banach left A-module with norms $\|.\|_X$ and $\|.\|_Y$, respectively. Let $U(A)$ be the set of unitary elements in A and let $r, s \in \mathbb{R}$ with $r+s, r-s \neq 0$. For a given mapping $f : X \to Y$, $u \in U(A)$ and a given $\mu \in \mathbb{C}$, we define $D_u f,\ D_\mu f : X^2 \to Y$ by

$$D_u f(x,y) := f(rux + suy) - \frac{r+s}{2} u f(x+y) - \frac{r-s}{2} u f(x-y),$$
$$D_\mu f(x,y) := f(r\mu x + s\mu y) - \frac{r+s}{2} \mu f(x+y) - \frac{r-s}{2} \mu f(x-y)$$

for all $x, y \in X$. An additive mapping $f : X \to Y$ is called A-linear if $f(ax) = af(x)$ for all $x \in X$ and all $a \in A$.

Proposition 1. *Let $L : X \to Y$ be a mapping with $L(0) = 0$ such that*

$$D_u L(x,y) = 0 \tag{2}$$

for all $x, y \in X$ and all $u \in U(A)$. Then L is A-linear.

Proof. Letting $y = x$ and $y = -x$ in (2), respectively, we get

$$L((r+s)ux) = \frac{r+s}{2} uL(2x), \qquad L((r-s)ux) = \frac{r-s}{2} uL(2x) \tag{3}$$

for all $x \in X$ and all $u \in U(A)$. Therefore it follows from (2) and (3) that

$$L(rux + suy) = L\big(\frac{r+s}{2} u(x+y)\big) + L\big(\frac{r-s}{2} u(x-y)\big) \tag{4}$$

for all $x, y \in X$ and all $u \in U(A)$. Replacing x by $\frac{1}{r+s}x + \frac{1}{r-s}y$ and y by $\frac{1}{r+s}x - \frac{1}{r-s}y$ in (4), we get

$$L(ux + uy) = L(ux) + L(uy) \tag{5}$$

for all $x, y \in X$ and all $u \in U(A)$. Hence L is additive (by letting $u = 1$ in (5)) and (3) implies that $L((r+s)ux) = (r+s)uL(x)$ for all $x \in X$ and all $u \in U(A)$. Since $r + s \neq 0$, we get

$$L(ux) = uL(x) \tag{6}$$

for all $x \in X$ and all $u \in U(A)$. It is clear that (6) holds for $u = 0$.

Now let $a \in A$ $(a \neq 0)$ and m an integer greater than $4|a|$. Then $|\frac{a}{m}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By Theorem 1 of [14], there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $\frac{3}{m}a = u_1 + u_2 + u_3$. So $a = \frac{m}{3}(\frac{3}{m}a) = \frac{m}{3}(u_1 + u_2 + u_3)$. Since L is additive, by (6) we have

$$\begin{aligned} L(ax) &= \frac{m}{3}L(u_1x + u_2x + u_3x) = \frac{m}{3}[L(u_1x) + L(u_2x) + L(u_3x)] \\ &= \frac{m}{3}(u_1 + u_2 + u_3)L(x) = \frac{m}{3}.\frac{3}{m}aL(x) = aL(x) \end{aligned}$$

for all $x \in X$. So $L : X \to Y$ is A-linear, as desired.

□

Corollary 2. *Let $L : X \to Y$ be a mapping with $L(0) = 0$ such that*

$$D_1L(x, y) = 0$$

for all $x, y \in X$. Then L is additive.

Corollary 3. *A mapping $L : X \to Y$ with $L(0) = 0$ satisfies*

$$D_\mu L(x, y) = 0$$

for all $x, y \in X$ and all $\mu \in \mathbb{T} := \{\, \mu \in \mathbb{C} : |\mu| = 1 \,\}$, if and only if L is $\mathbb{C}$-linear.

Now, we investigate the Hyers–Ulam–Rassias stability of the functional equation (1) in Banach modules.

We recall that throughout this paper $r, s \in \mathbb{R}$ with $r + s, r - s \neq 0$.

Theorem 4. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ such that*

$$\lim_{k\to\infty} \frac{1}{2^k}\varphi(2^kx, 2^ky) = 0, \tag{7}$$

$$\begin{aligned} \widetilde{\varphi}(x) := \sum_{k=0}^{\infty} \frac{1}{2^k}\Big\{\varphi\Big(\frac{2^{k+1}rx}{r^2 - s^2}, \frac{-2^{k+1}sx}{r^2 - s^2}\Big) \\ + \varphi\Big(\frac{2^kx}{r + s}, \frac{2^kx}{r + s}\Big) + \varphi\Big(\frac{2^kx}{r - s}, \frac{-2^kx}{r - s}\Big)\Big\} < \infty, \end{aligned} \tag{8}$$

$$\|D_1f(x, y)\|_Y \le \varphi(x, y) \tag{9}$$

for all $x, y \in X$. Then there exists a unique additive mapping $L : X \to Y$ such that

$$\|f(x) - L(x)\|_Y \le \frac{1}{2}\widetilde{\varphi}(x) \tag{10}$$

for all $x \in X$.

Proof. It follows from (9)that

$$\Big\|D_1f(x,y)-D_1f\big(\frac{x+y}{2},\frac{x+y}{2}\big)-D_1f\big(\frac{x-y}{2},\frac{y-x}{2}\big)\Big\|_Y \leq \varphi(x,y)+\varphi\big(\frac{x+y}{2},\frac{x+y}{2}\big)+\varphi\big(\frac{x-y}{2},\frac{y-x}{2}\big)$$

for all $x,y\in X$. Therefore

$$\begin{aligned}&\Big\|f(rx+sy)-f\big(\frac{r+s}{2}(x+y)\big)-f\big(\frac{r-s}{2}(x-y)\big)\Big\|_Y\\ &\leq \varphi(x,y)+\varphi\big(\frac{x+y}{2},\frac{x+y}{2}\big)+\varphi\big(\frac{x-y}{2},\frac{y-x}{2}\big)\end{aligned} \tag{11}$$

for all $x,y\in X$. Replacing x by $\frac{1}{r+s}x+\frac{1}{r-s}y$ and y by $\frac{1}{r+s}x-\frac{1}{r-s}y$ in (11), we get

$$\begin{aligned}\|f(x+y)-f(x)-f(y)\|_Y &\leq \varphi\big(\frac{x}{r+s}+\frac{y}{r-s},\frac{x}{r+s}-\frac{y}{r-s}\big)\\ &+\varphi\big(\frac{x}{r+s},\frac{x}{r+s}\big)+\varphi\big(\frac{y}{r-s},\frac{-y}{r-s}\big)\end{aligned} \tag{12}$$

for all $x,y\in X$. Letting $y=x$ in (12), we get

$$\begin{aligned}\|f(2x)-2f(x)\|_Y &\leq \varphi\big(\frac{2rx}{r^2-s^2},\frac{-2sx}{r^2-s^2}\big)\\ &+\varphi\big(\frac{x}{r+s},\frac{x}{r+s}\big)+\varphi\big(\frac{x}{r-s},\frac{-x}{r-s}\big)\end{aligned} \tag{13}$$

for all $x\in X$. For convenience, set

$$\psi(x):=\varphi\big(\frac{2rx}{r^2-s^2},\frac{-2sx}{r^2-s^2}\big)+\varphi\big(\frac{x}{r+s},\frac{x}{r+s}\big)+\varphi\big(\frac{x}{r-s},\frac{-x}{r-s}\big)$$

for all $x\in X$. It follows from (8) that

$$\sum_{k=0}^{\infty}\frac{1}{2^k}\psi(2^kx)=\widetilde{\varphi}(x)<\infty \tag{14}$$

for all $x\in X$. Replacing x by 2^kx in (13) and dividing both sides of (13) by 2^{k+1}, we get

$$\Big\|\frac{1}{2^{k+1}}f(2^{k+1}x)-\frac{1}{2^k}f(2^kx)\Big\|_Y \leq \frac{1}{2^{k+1}}\psi(2^kx)$$

for all $x\in X$ and all $k\in\mathbb{N}$. Therefore we have

$$\begin{aligned}\Big\|\frac{1}{2^{k+1}}f(2^{k+1}x)-\frac{1}{2^m}f(2^mx)\Big\|_Y &\leq \sum_{l=m}^{k}\Big\|\frac{1}{2^{l+1}}f(2^{l+1}x)-\frac{1}{2^l}f(2^lx)\Big\|_Y\\ &\leq \frac{1}{2}\sum_{l=m}^{k}\frac{1}{2^l}\psi(2^lx)\end{aligned} \tag{15}$$

for all $x \in X$ and all integers $k \geq m \geq 0$. It follows from (14) and (15) that the sequence $\{\frac{f(2^k x)}{2^k}\}$ is a Cauchy sequence in Y for all $x \in X$, and thus converges by the completeness of Y. So we can define the mapping $L : X \to Y$ by

$$L(x) = \lim_{k\to\infty} \frac{f(2^k x)}{2^k}$$

for all $x \in X$. Letting $m = 0$ in (15) and taking the limit as $k \to \infty$ in (15), we obtain the desired inequality (10). It follows from the definition of L, (7) and (9) that

$$\begin{aligned}\|D_1 L(x,y)\|_Y &= \lim_{k\to\infty} \frac{1}{2^k}\|D_1 f(2^k x, 2^k y)\|_Y \\ &\leq \lim_{k\to\infty} \frac{1}{2^k}\varphi(2^k x, 2^k y) = 0\end{aligned}$$

for all $x, y \in X$. Therefore the mapping $L : X \to Y$ satisfies the equation (1) and $L(0) = 0$. Hence by Corollary 2, L is a additive mapping.

To prove the uniqueness of L, let $L' : X \to Y$ be another additive mapping satisfying (10). Therefore it follows from (10) and (14) that

$$\begin{aligned}\|L(x) - L'(x)\|_Y &= \lim_{k\to\infty} \frac{1}{2^k}\big\|f(2^k x) - L'(2^k x)\big\|_Y \\ &\leq \frac{1}{2}\lim_{k\to\infty} \frac{1}{2^k}\sum_{l=0}^{\infty} \frac{1}{2^l}\psi(2^{l+k} x) \\ &= \frac{1}{2}\lim_{k\to\infty} \sum_{l=k}^{\infty} \frac{1}{2^l}\psi(2^l x) = 0\end{aligned}$$

for all $x \in X$. So $L(x) = L'(x)$ for all $x \in X$. It completes the proof. □

Theorem 5. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (7), (8) and*

$$\|D_u f(x,y)\| \leq \varphi(x,y)$$

for all $x, y \in X$ and all $u \in U(A)$. Then there exists a unique A-linear mapping $L : X \to Y$ satisfying (10) for all $x \in X$.

Proof. By Theorem 4 (letting $u = 1$), there exists a unique additive mapping $L : X \to Y$ satisfying (10) and

$$L(x) = \lim_{k\to\infty} \frac{f(2^k x)}{2^k}$$

for all $x \in X$. By the assumption, we have

$$\begin{aligned}\big\|D_u L(x,y)\big\|_Y &= \lim_{k\to\infty} \frac{1}{2^k}\big\|D_u f(2^k x, 2^k y)\big\|_Y \\ &\le \lim_{k\to\infty} \frac{1}{2^k}\varphi(2^k x, 2^k y) = 0\end{aligned}$$

for all $x, y \in X$ and all $u \in U(A)$. Since $L(0) = 0$, by Proposition 1 the additive mapping $L : X \to Y$ is A-linear. $\square$

Corollary 6. *Let δ, ε, p and q be non-negative real numbers such that $0 < p, q < 1$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{aligned}&\|D_1 f(x,y)\|_Y \le \delta + \varepsilon(\|x\|_X^p + \|y\|_X^q)\\ &\Big(\|D_u f(x,y)\|_Y \le \delta + \varepsilon(\|x\|_X^p + \|y\|_X^q)\Big)\end{aligned}$$

for all $x, y \in X$ (and all $u \in U(A)$). Then there exists a unique additive (A-linear) mapping $L : X \to Y$ such that

$$\begin{aligned}\|f(x) - L(x)\|_Y \le 3\delta &+ \frac{2|r|^p + |r+s|^p + |r-s|^p}{(2-2^p)|r^2-s^2|^p}\varepsilon\|x\|_X^p \\ &+ \frac{2|s|^q + |r+s|^q + |r-s|^q}{(2-2^q)|r^2-s^2|^q}\varepsilon\|x\|_X^q\end{aligned} \tag{16}$$

for all $x \in X$.

Proof. Define $\varphi(x,y) := \delta + \varepsilon(\|x\|_X^p + \|y\|_X^q)$, and apply Theorem 4 (Theorem 5). $\square$

Remark 7. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\Phi : X^2 \to [0,\infty)$ satisfying

$$\lim_{n\to\infty} 2^n \Phi\big(\frac{x}{2^n}, \frac{y}{2^n}\big) = 0, \tag{17}$$

$$\begin{aligned}\widetilde{\Phi}(x) := \sum_{k=1}^{\infty} 2^k \Big\{ &\Phi\big(\frac{2rx}{2^k(r^2-s^2)}, \frac{-2sx}{2^k(r^2-s^2)}\big) + \Phi\big(\frac{x}{2^k(r+s)}, \frac{x}{2^k(r+s)}\big) \\ &+ \Phi\big(\frac{x}{2^k(r-s)}, \frac{-x}{2^k(r-s)}\big)\Big\} < \infty,\end{aligned} \tag{18}$$

$$\|D_1 f(x,y)\| \le \Phi(x,y) \quad \Big(\|D_u f(x,y)\| \le \Phi(x,y)\Big)$$

for all $x, y \in X$ (and all $a \in U(A)$). By a similar method to the proof of Theorem 4, one can show that there exists a unique additive (A-linear) mapping $L : X \to Y$ satisfying

$$\|f(x) - L(x)\| \le \frac{1}{2}\widetilde{\Phi}(x)$$

for all $x \in X$.

For the case $\Phi(x,y) := \varepsilon(\|x\|^p + \|y\|^q)$ (where ε, p and q are non-negative real numbers with $p, q > 1$), there exists a unique additive (A-linear) mapping $L : X \to Y$ satisfying

$$\begin{aligned} \|f(x) - L(x)\|_Y \leq & \frac{2|r|^p + |r+s|^p + |r-s|^p}{(2^p-2)|r^2-s^2|^p}\varepsilon\|x\|_X^p \\ & + \frac{2|s|^q + |r+s|^q + |r-s|^q}{(2^q-2)|r^2-s^2|^q}\varepsilon\|x\|_X^q \end{aligned} \tag{19}$$

for all $x \in X$.

Corollary 8. *Let ε, p and $q > 0$ be non-negative real numbers such that $\lambda := p+q \neq 1$ and $|r| \neq |r|^\lambda$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_1 f(x,y)\|_Y \leq \varepsilon\|x\|_X^p\|y\|_X^q$$
$$\Big(\|D_u f(x,y)\|_Y \leq \varepsilon\|x\|_X^p + \|y\|_X^q\Big)$$

for all $x, y \in X$ (and all $u \in U(A)$). Then f is additive (A-linear).

3. Homomorphisms between C^*-algebras

Homomorphisms between C^*-algebrasThroughout this section, assume that A is a unital C^*-algebra with norm $\|.\|_A$ and B is a C^*-algebra with norm $\|.\|_B$. We recall that throughout this paper $r, s \in \mathbb{R}$ with $r+s, r-s \neq 0$.

We investigate C^*-algebra homomorphisms between C^*-algebras.

Theorem 9. *Let $f : A \to B$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (7), (8) and*

$$\|D_\mu f(x,y)\|_B \leq \varphi(x,y), \tag{20}$$
$$\|f(2^k u^*) - f(2^k u)^*\|_B \leq \varphi(2^k u, 2^k u), \tag{21}$$
$$\|f(2^k ux) - f(2^k u)f(x)\|_B \leq \varphi(2^k ux, 2^k ux) \tag{22}$$

for all $x, y \in A$, all $u \in U(A)$, all $\mu \in \mathbb{S}^1$ and all $k \in \mathbb{N}$. Then there exists a unique C^*-algebra homomorphism $H : A \to B$ satisfying (10) for all $x \in X$. Moreover $H(x)[H(y) - f(y)] = 0$ for all $x, y \in A$.

Proof. By the same reasoning as in the proof of Theorem 5, there exists a unique $\mathbb{C}$-linear mapping $H : A \to B$ satisfying (10). The mapping $H : A \to B$ is defined by

$$H(x) = \lim_{k\to\infty} \frac{1}{2^k} f(2^k x)$$

for all $x \in A$. Hence it follows from (7), (21) and (22) that

$$\begin{aligned}\|H(u^*) - H(u)^*\|_B &= \lim_{k\to\infty} \frac{1}{2^k}\|f(2^k u^*) - f(2^k u)^*\|_B \\ &\le \lim_{k\to\infty} \frac{1}{2^k}\varphi(2^k u, 2^k u) = 0,\end{aligned}$$

$$\begin{aligned}\|H(ux) - H(u)f(x)\|_B &= \lim_{k\to\infty} \frac{1}{2^k}\|f(2^k ux) - f(2^k u)f(x)\|_B \\ &\le \lim_{k\to\infty} \frac{1}{2^k}\varphi(2^k ux, 2^k ux) = 0\end{aligned}$$

for all $x \in A$ and all $u \in U(A)$. So $H(u^*) = H(u)^*$ and $H(ux) = H(u)f(x)$ for all $x \in A$ and all $u \in U(A)$. Since H is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [13]), i.e., $x = \sum_{k=1}^m \lambda_k u_k$, where $\lambda_k \in \mathbb{C}$ and $u_k \in U(A)$ for all $1 \le k \le n$, we have

$$\begin{aligned}H(x^*) &= H\big(\sum_{k=1}^m \overline{\lambda_k} u_k^*\big) = \sum_{k=1}^m \overline{\lambda_k} H(u_k^*) = \sum_{k=1}^m \overline{\lambda_k} H(u_k)^* \\ &= \big(\sum_{k=1}^m \lambda_k H(u_k)\big)^* = \big[H\big(\sum_{k=1}^m \lambda_k u_k\big)\big]^* = H(x)^*,\end{aligned}$$

$$\begin{aligned}H(xy) &= H\big(\sum_{k=1}^m \lambda_k u_k y\big) = \sum_{k=1}^m \lambda_k H(u_k y) \\ &= \sum_{k=1}^m \lambda_k H(u_k) f(y) = H\big(\sum_{k=1}^m \lambda_k u_k\big) f(y) = H(x)f(y)\end{aligned}$$

for all $x, y \in A$. Since H is $\mathbb{C}$-linear, we have

$$H(xy) = \lim_{k\to\infty} \frac{1}{2^k} H(2^k xy) = \lim_{k\to\infty} \frac{1}{2^k} H(x)f(2^k y) = H(x)H(y)$$

for all $x, y \in A$. Therefore the mapping $H : A \to B$ is a C^*-algebra homomorphism and $H(x)[H(y) - f(y)] = 0$ for all $x, y \in A$. □

Corollary 10. *Let δ, ε, p and q be non-negative real numbers such that $0 < p, q < 1$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\begin{aligned}\|D_\mu f(x, y)\|_Y &\le \delta + \varepsilon(\|x\|_X^p + \|y\|_X^q) \\ \|f(2^k u^*) - f(2^k u)^*\|_B &\le \delta + \varepsilon(2^{kp} + 2^{kq}), \\ \|f(2^k ux) - f(2^k u)f(x)\|_B &\le \delta + \varepsilon(2^{kp}\|x\|_X^p + 2^{kq}\|x\|_X^q)\end{aligned}$$

for all $x, y \in A$, all $u \in U(A)$, all $\mu \in \mathbb{S}^1$ and all $k \in \mathbb{N}$. Then there exists a unique C^*-algebra homomorphism $H : A \to B$ satisfying (16) for all $x \in X$. Moreover

$$H(x)[H(y) - f(y)] = 0$$

for all $x, y \in A$.

Remark 11. Let $f : X \to Y$ be a mapping with $f(0) = 0$ for which there exists a function $\Phi : X^2 \to [0, \infty)$ satisfying (17), (18) and

$$\|D_\mu f(x, y)\|_B \le \Phi(x, y),$$
$$\|f(\frac{u^*}{2^k}) - f(\frac{u}{2^k})^*\|_B \le \Phi(\frac{u}{2^k}, \frac{u}{2^k}),$$
$$\|f(\frac{ux}{2^k}) - f(\frac{u}{2^k})f(x)\|_B \le \Phi(\frac{ux}{2^k}, \frac{ux}{2^k})$$

for all $x, y \in A$, all $u \in U(A)$, all $\mu \in \mathbb{S}^1$ and all $k \in \mathbb{N}$. By a similar method to the proof of Theorem 9, one can show that there exists a unique C^*-algebra homomorphism $H : A \to B$ satisfying (10) and $H(x)[H(y) - f(y)] = 0$ for all $x, y \in A$.

For the case $\Phi(x, y) := \varepsilon(\|x\|^p + \|y\|^q)$ (where ε, p and q are non-negative real numbers with $p, q > 1$), there exists a unique C^*-algebra homomorphism $H : A \to B$ satisfying (19) and $H(x)[H(y) - f(y)] = 0$ for all $x, y \in A$.

Applying Corollary 8, Theorem 9 and Remark 11, we get the following results.

Theorem 12. *Let ε, p and $q > 0$ be non-negative real numbers such that $\lambda := p+q < 1$ and $|r| \ne |r|^\lambda$. Let $f : A \to B$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\varphi : X^2 \to [0, \infty)$ satisfying (7) and*

$$\|D_\mu f(x, y)\|_Y \le \varepsilon \|x\|_X^p \|y\|_X^q$$
$$\|f(2^k u^*) - f(2^k u)^*\|_B \le \varphi(2^k u, 2^k u),$$
$$\|f(2^k ux) - f(2^k u)f(x)\|_B \le \varphi(2^k ux, 2^k ux)$$

for all $x, y \in A$, all $u \in U(A)$, all $\mu \in \mathbb{S}^1$ and all $k \in \mathbb{N}$. Then f is a C^*-algebra homomorphism.

Theorem 13. *Let ε, p and $q > 0$ be non-negative real numbers such that $\lambda := p+q > 1$ and $|r| \ne |r|^\lambda$. Let $f : A \to B$ be a mapping satisfying $f(0) = 0$ for which there exists a function $\Phi : X^2 \to [0, \infty)$ satisfying (17) and*

$$\|D_\mu f(x, y)\|_Y \le \varepsilon \|x\|_X^p \|y\|_X^q$$
$$\|f(\frac{u^*}{2^k}) - f(\frac{u}{2^k})^*\|_B \le \Phi(\frac{u}{2^k}, \frac{u}{2^k}),$$
$$\|f(\frac{ux}{2^k}) - f(\frac{u}{2^k})f(x)\|_B \le \Phi(\frac{ux}{2^k}, \frac{ux}{2^k})$$

for all $x, y \in A$, all $u \in U(A)$, all $\mu \in \mathbb{S}^1$ and all $k \in \mathbb{N}$. Then f is a C^*-algebra homomorphism.

Corollary 14. *Let ε, p and $q > 0$ be non-negative real numbers such that $\lambda := p + q \neq 1$ and $|r| \neq |r|^\lambda$. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the inequality*

$$\|D_\mu f(x,y)\|_Y \leq \varepsilon \|x\|_X^p \|y\|_X^q$$
$$\|f(2^k u^*) - f(2^k u)^*\|_B \leq \varepsilon 2^{k\lambda},$$
$$\|f(2^k ux) - f(2^k u)f(x)\|_B \leq \varepsilon 2^{k\lambda} \|x\|_X^{k\lambda}$$

for all $x, y \in A$, all $u \in U(A)$, all $\mu \in \mathbb{S}^1$ and all $k \in \mathbb{Z}$. Then f is a C^*-algebra homomorphism.

References

[1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables,* Cambridge University Press, 1989.

[2] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[3] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach,* Grazer Math. Ber. **346** (2004), 43–52.

[4] P.W. Cholewa, *Remarks on the stability of functional equations,* Aequationes Math. **27** (1984), 76–86.

[5] S. Czerwik, *On the stability of the quadratic mapping in normed spaces,* Abh. Math. Sem. Univ. Hamburg **62** (1992), 59–64.

[6] S. Czerwik, *Functional Equations and Inequalities in Several Variables*, World Scientific Publishing Company, New Jersey, Hong Kong, Singapore and London, 2002.

[7] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. **184** (1994), 431–436.

[8] A. Grabiec, *The generalized Hyers–Ulam stability of a class of functional equations,* Publ. Math. Debrecen **48** (1996), 217–235.

[9] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941), 222–224.

[10] D.H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[11] G. Isac and Th. M. Rassias, *Stability of Ψ-additive mappings: applications to nonlinear analysis*, Internat. J. Math. Math. Sci. **19** (1996), 219–228.

[12] S.-M. Jung, *Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press lnc., Palm Harbor, Florida, 2001.

[13] R.V. Kadison and J.R. Ringrose, *Fundamentals of the Theory of Operator Algebras*, Academic Press, New York, 1983.

[14] R.V. Kadison and G. Pedersen, *Means and convex combinations of unitary operators,* Math. Scand. **57** (1985), 249–266.

[15] M.S. Moslehian and Th.M. Rassias, *Stability of functional equations in non-Archimedian spaces*, Appl. Anal. Disc. Math., **1** (2007), 325–334.

[16] A. Najati, *Hyers–Ulam stability of an n-Apollonius type quadratic mapping,* Bull. Belgian Math. Soc.-Simon Stevin **14** (2007), 755–774.

[17] A. Najati, *On the stability of a quartic functional equation,* J. Math. Anal. Appl. **340** (2008), 569–574.

[18] A. Najati and M.B. Moghimi, *Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces,* J. Math. Anal. Appl. **337** (2008), 399–415.

[19] A. Najati and C. Park, *Hyers–Ulam–Rassias stability of homomorphisms in quasi-Banach algebras associated to the Pexiderized Cauchy functional equation,* J. Math. Anal. Appl. **335** (2007), 763–778.

[20] A. Najati and C. Park, *The Pexiderized Apollonius-Jensen type additive mapping and isomorphisms between C^*-algebras,* J. Difference Equat. Appl. **14** (2008), 459–479.

[21] C. Park, *On the stability of the linear mapping in Banach modules,* J. Math. Anal. Appl. **275** (2002), 711–720.

[22] C. Park, *Lie $*$-homomorphisms between Lie C^*-algebras and Lie $*$-derivations on Lie C^*-algebras*, J. Math. Anal. Appl. **293** (2004), 419–434.

[23] C. Park, *Homomorphisms between Poisson JC^*-algebras*, Bull. Braz. Math. Soc. **36** (2005), 79–97.

[24] C. Park, *Hyers–Ulam–Rassias stability of a generalized Euler–Lagrange type additive mapping and isomorphisims between C^*-algebras*, Bull. Belgian Math. Soc.–Simon Stevin **13** (2006), 619–631.

[25] C. Park and A. Najati, *Homomorphisms and derivations in $C*$-algebras,* Abstract and Applied Analysis, (2007) Article ID 80630, 12 pages.

[26] J.M. Rassias, *On approximation of approximately linear mappings by linear mappings*, Bull. Sci. Math. **108** (1984), 445–446.

[27] J.M. Rassias, *Solution of a problem of Ulam*, J. Approx. Theory **57** (1989), 268–273.

[28] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[29] Th.M. Rassias, *The problem of S.M. Ulam for approximately multiplicative*

mappings, J. Math. Anal. Appl. **246** (2000), 352–378.

[30] Th.M. Rassias, *On the stability of functional equations and a problem of Ulam*, Acta Appl. Math. **62** (2000), 23–130.

[31] Th.M. Rassias, *Functional Equations, Inequalities and Applications*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2003.

[32] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.

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