CALCULATING THE NUMBER OF THE LINEAR FACTORS IN PLANAR GRAPHS

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Abstract. This research work aims to calculate the number of the linear factors and paths in a binary bipartite graph, using kasteleyn theorem additionally, it helps in defining a new directing χ, using pfaff directing.

Keywords: Path, pfaff, directed graph, bipartite graph, planar graph, tree.

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1. Introduction

In this research work I intend to present some basic concepts in relation to this research to point to the work's importance in applied mathematics in general and chemistry and physics in particular.

The problem of covering in chemical properties for double coupling that covers a crystal face, comes to the surface ; a case that could be simulated by quadratic netword F. This problem means that every two vertix in the graph F are connected.

The complete covering D of the graph F means that every vertix in F is totally covered, as shown in the following figure :

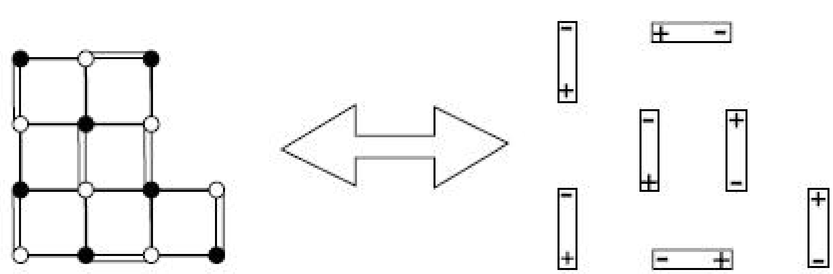


Fig 1.

The following two poin's are the most important for physicians :

1- The number of the complete coverings for graph F means the number of the linear factors $m(F)$ of F.

2- The number of coverings for two connected vertix in the graph F, that means the number of linear factors $m(F, e)$ for graph F will contain the edge: $e = (x, y)$.

All complete coverings can be considired as probable equation :

$$P(F, e) = m(F, e)/m(F) \tag{1}$$

This problem has been considired by M. E. Fisherm 1961, in order to solve the problem of coverings for a section in a quadratic network.

$$F = [1, p] \times [1, q] \qquad (pq = 0,\ mod2)$$

This problem appears in organic chemistry, especially in hexaconal system for carbon compounds. Hexaconal system is a binary planar graph connected and finite.

All areas in this graph will be regular hexaconals, where the lenght of each edge is 1.

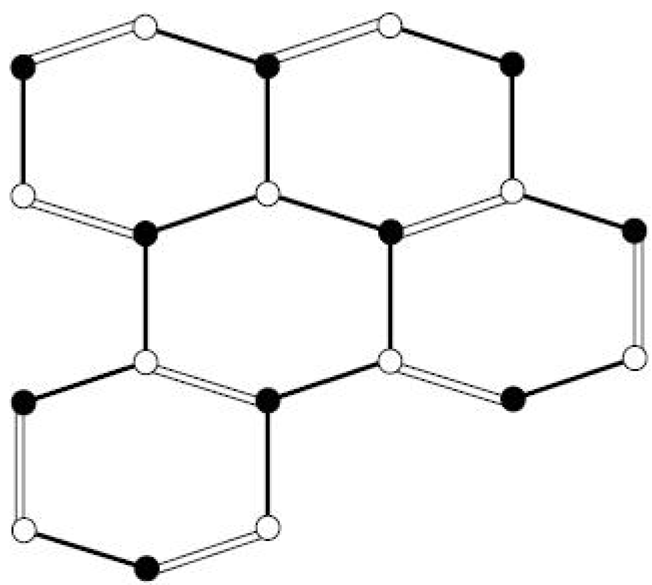


Fig 2.

In this case, it is important to complete $m(H)$ and $m(H, e)$, to find the probability :

$$P(H, e) = m(H, e)/m(H) \tag{2}$$

This is concordant with the conditions set up by pauling.

There are algorithms to compute $m(H)$ and pauling conditions $p(e)$ for carbon compounds (H. Sachs , P. John [6]).

We will compute the number of linear factors in some planar graphs, taking as a starting point (P. W. Kasteleyn [7]).

2.Definitions and theorems

We present hereunder the development of P. W. Kasteleyn theorem and its applications on planar graphs, using pfaff direction (C. H. C. Little [9]).

These graphics including Izomorphic sub-graph with graph $K_{3.3}$(C. H. C. Little).

Kasteleyn direction :

Let $G = (Z.E)$ be simple connected planar graph, and $\underline{G}$ is the projection of G in the plane. Let W be a direction of G, where every edge from E has a direction, that is, the edge $(x, y) \in E$ takes the arc (x, y) or (y, x) ; as such, we denote the directed graph with the following : $\overrightarrow{G}^W = (Z, \overrightarrow{E}^W)$.

We refer to W as kasteleyn direction of graph $\underline{G}$ if and only if every area in G (and all infinite area), where the number of the directed left edges is odd. Figure (3).

We denote to kasteleyn directing with χ.

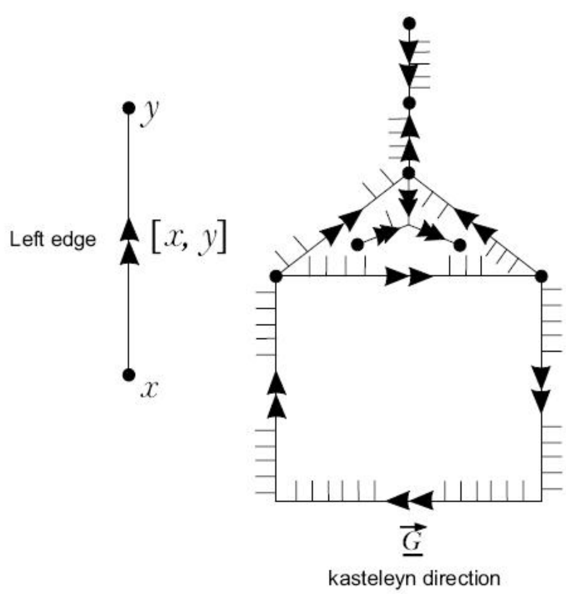


Fig 3.

Adjacent matrices of the following two graphs G, $\overrightarrow{G}^w$:
(i) Adjacent matrix A for the graph G :

$$A = (a_{ik});\;\; a_{ik} = \begin{cases} 1 : (x_i, x_k) \in E.....(3) \\ 0 : (x_i, x_k) \notin E \end{cases}$$

(ii) Adjacent matrix A^w for the graph $\overrightarrow{G}^w$:

$$A^W = (a_{ik}^W);\;\; a_{ik}^W = \begin{cases} 1 : (x_i, x_k) \in \overrightarrow{E}^W \\ 0 : (x_i, x_k) \notin \overrightarrow{E}^W \end{cases}$$

(iii) The developed adjacent matrix of the graph $\overrightarrow{G}^w$ is :

$$S^W = (S_{ik}^W);\;\; S_{ik}^W = \begin{cases} 1 : (x_i, x_k) \in \overrightarrow{E}^W \\ -1 : (x_k, x_i) \in \overrightarrow{E}^W \\ 0 : (x_i, x_k) \notin \overrightarrow{E}^W \end{cases}$$

Thus:

$$\left|S_{ik}^W\right| = a_{ik}, \quad A = A^W + A^{WT}, \quad S^W = A^W - A^{WT} \tag{3}$$

A symetric matrix:

$$A^T = A \tag{4}$$

S^W Alternative symmetrical matrix:

$$S^{WT} = -S^W \tag{5}$$

Then we have the following equation :

$$\det A = -\det A^T \tag{6}$$

Theorem 1 (Kasteleyn Theorem). *Let G be a simple connected graph in the plan, and let $\underline{G}$ be the projection of the graph G in the plan, and let χ be Kasteleyn directing for $\underline{G}$ and Let m be the number of the linear coefficients of the graph G then:*

$$m^2 = \det S^\chi \tag{7}$$

We will use the previous theorem to develop the algorithm that calculates the number of the linear factors in a planar graph.

Not1: Each connected planar graph with even number of vertices has kasteleyn directing, such a directing can be easily obtained.

Theorem 2. *Let G be a connected planar graph χ with n vertices then:*

1- *G has Kasteleyn directing if and only if n was an even number.*

2- *An algorithm to construct Kasteleyn directing can be found as follows:*

We search for a tree $\underline{B}$ in the graph $\underline{G}$, we direct the edges of the tree $\underline{B}$ optionally, then we odd the edges of the graph $\underline{G}$ which are not in the tree $\underline{B}$, we direct the added edges in away that the directing of the structured graph is kasteleyn directing, we can do that in exactly one way.

3- *$\underline{G}$ has 2^{n-1} Kasteleyn directing, corresponding of the tree $\underline{B}$.*

Definition 1. *We call the edge of G a cycle edge if it is contained in one cycle in $\underline{G}$ at least.*

Lemma 1. *Let $\underline{e}$ be a cycle edge in $\underline{G}$ and let $\underline{G}^{-1} = \underline{G} - \underline{e}$ then:*

(i)- *If χ was a kasteleyn directing of $\underline{G}$ then the contracted χ' of χ on $\underline{G}'$ is a kasteleyn directing.*

(ii)- *If χ' was a kasteleyn directing of $\underline{G}'$ then the directing will be considered uniquely , in order to be χ for χ' a kasteleyn directing for $\underline{G}$.*

Proof of Theorem 2. Using mathematical induction and applying lemma 1 proove theorem 2.

Example 1. Let's take the planar graph $G_2^0 = (Z^0, E^0)$ with $Z^0 = \{Z_1, Z_2, Z_3, Z_4\}$ and $E^0 = \{(Z_1, Z_2), (Z_1, Z_3), (Z_2, Z_3), (Z_3, Z_4)\}$

For example:

The two graphs in (Figure 4) give two projections: $\underline{G}_1^0$, $\underline{G}_2^0$ of G^0. But both of them has 8 kasteleyn directings, i.e.16 directings, and these directings are binarily different, which means that W directing of $\underline{G}^0$ for the projection G^0 in the plane is a kasteleyn directing which means that formula (7) is true.

If we has a kasteleyn directing χ then we may put the inversion $-\chi$ in the formula (7) and it remains true, because $-\chi$ is a kasteleyn directing.

Let's simplify kasteleyn theorem for the bipartite planar graphs.

The bipartite graph $G_2 = (Z, E) = (X, Y, E)$ has a linear factors if and only if $|x| = |y|$, let's assume that the previous conditions holds.

If we labeled the vertices, then we will have :

$$A = \begin{pmatrix} O & B \\ B^T & O \end{pmatrix} \tag{8}$$

and

$$S^\chi = \begin{pmatrix} O & U^\chi \\ -U^{\chi T} & O \end{pmatrix} \tag{9}$$

where

$$B = (b_{ik}) \qquad (b_{ik} \in \{0.1\})$$

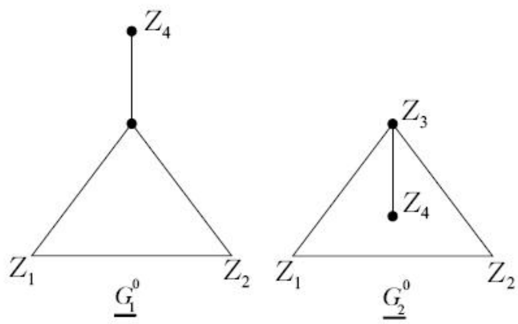


Fig 4.

and

$$U^{\chi} = (u^{\chi}_{ik}) \quad (u^{\chi}_{ik} \in \{-1.0, 1\}).$$

Squared Matrices of dimension $|x| = |y| = n/2$, and if we applied in the formula (7) then:

$$m = |\det U^{\chi}| \tag{10}$$

The graph $\underline{G}$ has the property that each area R in $\underline{G}$ has an odd number of left edges which belongs to R, so, we have a kasteleyn directing. In particular, we direct all the edges from Y to X then $u^{\chi} = -B$, doing that gives us the following formula:

$$m = |\det B| \tag{11}$$

We can prove formulas (10) and (11) using pfaff's concept.

So, we simplified kasteleyn theorem to calculate the number of linear factors in bipartite planar graphs.

3.The number of linear factors in bipartite planar graphs

We will present the concept of finite bipartite planar graphs.

Suppose that $|X| = |Y|$, because else than this there doesn't exist a linear factor. And let's also suppose that $|X| = n/2$ and G is a connected graph.

Let $\underline{G} = (Z, E) = (X, Y, E)$ be the projection of the graph G in the plane, in order to calculate the number of linear factors in G, we need to find a kasteleyn directing of graph $\underline{G}$, then calculate the determinant of the matrix u^{χ} (or B) using the algorithm of calculating the determinate of a matrix then we obtain the number of linear factors in G.

Let $e = (x_i, y_k) \in E$ Then:

$$u_{ik}^{\chi} = \begin{cases} -1 \text{ if } : [y_k; x_i] \in \overrightarrow{E}^{\chi} \\ +1 \text{ if } : [x_i, y_k] \in \overrightarrow{E}^{\chi} \\ 0 \text{ if } : [x_i, y_k] \notin \overrightarrow{E}^{\chi} \end{cases} \tag{12}$$

The algorithm of calculating the number of linear factors of bipartite graph:

1 - We provide every top vertex with an identity vector of a dimension equals to the number of tops .

2 - The vectors. In the black vertices, we odd the vectors of the incident vertices multiplied with the weight of vertices.

3 - In the white vertices divided by the weight of the matching then we change the signal.

4 - We order the vectors of the tails in a matrix and calculate its determinants.

Note 2: The value of the determinant represents the number of the linear factors.

4. A method to calculate the number of linear factors in bipartite graphs

To calculate the determinant of the matrix U^{χ} we refer to every row of U^{χ} with a black vertex and to every column with a white vertix , then we obtain the graph G^{χ}. if we didn't consider the weight of the edges $-1, +1$ then this graph is isomorphic with the graph G.

We apply the previous algorithm of calculating the determinant of a matrix on the graph G^{χ} or directly on G, considering the weight of edges $-1, +1$.

Example 2. (Problem of covering aches board) A chess board $S_{5,4}$ can be covered by ten domino stones (every two incident vertices are covered), such that every field is covered with one stone only.

Let $\underline{G}$ be the graph resulting of $S_{5,4}$ (See figure 5) i.e, calculating the number of linear factors m of graph $\underline{G}$.

The result will be the calculation of the number of linear factors: M is a cycle matching $q(u, v) \ldots..(u \in U, v \in V)$.

$$m = abs \begin{vmatrix} 16 & 23 \\ 7 & 16 \end{vmatrix} = 95.$$

So, we can cover $S_{5,4}$ in 95 ways using ten domino's stones.

In the same way, we can calculate $m(\underline{G}, e)$ which represents the number of linear factors that doesn't contain the edge $e = (a, b)$ (figure 5).

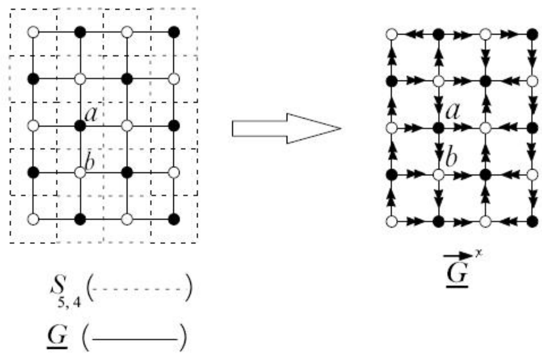


Fig 5.

Here, we will discuss special cases for the graph G with $\underline{G}$ as a projection, in which the number of edges in area R (the number of edges that their left edges belong to R) is odd.

$U^{\chi} = B(i.e.G^{\chi} = G)$, we will use formula (11) for that.

Example 3. (A special case of carbonic)

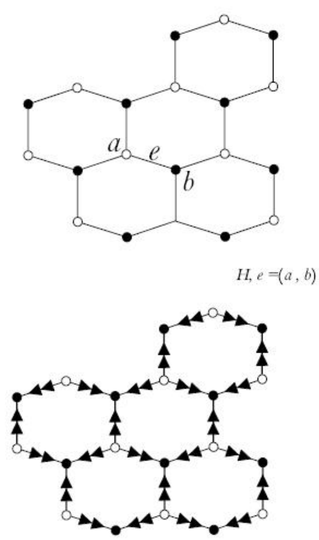
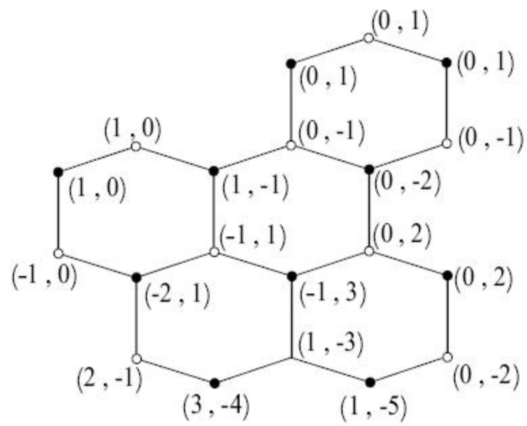


Fig 6.



$$m = abs \begin{vmatrix} 3 & -4 \\ 1 & -5 \end{vmatrix} = 11$$

Fig 7.

Or using another kasteleyn directing, from up to down.

Definition of Tops and Tails:

- The white vertices which are not incident to a matching are called tops.

- The black vertices which are incidents to a matching are called tails.

Note 4: The coefficient in the tail's vectors represents the number of the successive paths in the graphs which connect the tops with the tails U.

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