ON SPECIFICITY AND ENTROPY MEASURES

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Abstract. Our objective in this paper will be contribute to develop certain useful analytical tools on the foundations of Uncertainty Measures. Because we need to obtain new ways to model adequate conditions or restrictions, constructed from vague pieces of information. For this, it is necessary to classify more efficiently the distinct types of measures; in particular, the fuzzy measures.

We made a first approximation as Plenary Speaker in the *ICMI 45,* at Bacau (2006), revisiting such problem in both precedent *ICTAMIs* (very recommendable series of Conferences, organized at Alba Iulia University), in concrete the celebrated in the years 2007 and 2009.

Now, we complete this study by the analysis on Specificity as a very interesting Measure of Uncertainty, with their mutual relationships with Entropy.

So, we attempt to go on, advancing by this paper.

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1.Introduction

Taking the Entropy concept, we attempt to measure the fuzziness, that is, the degree of fuzziness for each element

$$A \in \wp$$

It can be designed as the function

$$H : \wp \to [0, 1]$$

Verifying that

I) If A is a *crisp set,* then $H(A) = 0$.

II) If $H(x)=1/2$, for each x in A, then $H(A)$ *is maximal* (i.e. the maximal uncertainty state).

III) If A *is less fuzzified than* B, then

$$H(A) \leq H(B)$$

IV)$H(A) = H(U \backslash A)$.

Given a discrete random variable, *X,* with an associated discrete probability distribution *P(x),* we may define the *Entropy (H) of X* as

$$H\left(x\right) \equiv -\sum_{x \in X} p\left(x\right) \ln p\left(x\right) = \sum_{x \in X} p\left(x\right) \ln\left(1/p\left(x\right)\right) = E\left[\ln \frac{1}{p\left(x\right)}\right]$$

And in the continuous case,

$$H\left(x\right) \equiv -\int p\left(x\right) \ln p\left(x\right)\ dx = \int p\left(x\right) \ln\left(1/p\left(x\right)\right)\ dx = E\left[\ln \frac{1}{p\left(x\right)}\right]$$

Such H is a measure of the quantity of information that we receipt, when is sended towards us.

The logaritmic base will be arbitrary.

If *b = 2,* it is measured in *bits.*

If *b = 10,* it will be in *dits.*

And if *b = e,* in *nats.*

We may see now the *Entropy Measures* in a more generalized version.

Departing of a *t-norm, T,* a *t-conorm, S,* and the negation, *N,* it will be possible to introduce the *Entropy Measure* in another different way.

We have the subsequent *results*

1) If m is a crisp set, then *H (m) = 0.*

2) When T is in the minimum in the family of the t-norm product, then

$H(m) = 0$ if and only if m *is a crisp set*
3) If A is *less fuzzified than B,* then

$$H(A) \leq H(B).$$

4)$H(A) = H(U \backslash A)$.

5) $H(m) = H(c(m))$.

In this way, we may found that the Entropy Measure holds the properties signaled by *De Luca and Termini,* in 1972.

Belis and Guiasu, in 1968, introduces a *general "weighted" Shannon Entropy,* by

$$H_{BGE}(x) \equiv - \sum_{i=1,2,\ldots,n} w_i \; p_i \; \ln p_i$$

being $\{w_i\}_{i=1}^n$ a collection of non-negative coefficients, which denotes the subjective importance of the distinct possible outcomes of the corresponding probabilistic distribution.

Both authors have also suggested that the occurrence of each event gives us two types of uncertainty,

- the *qualitative type,* related to its utility

- the *quantitative type,* relative to its probability of occurrence

2.Specificity measure

The introduction of measures of tranquillity, by *Yager* (1990) will be useful taking decisions. Later, also Yager introduce the *"specificity-correctness trad-off principle".*

Dubois and Prade (1999) give the *minimal specificity principle,* showing the role of specificity in Approximate Reasoning.

Higashi and Klir (1983), introduces a very related idea, the *non-specificity concept.*

Previously, the concept of *granularity,* by *Zadeh* (1978) also maintain a close relation with them.

Zadeh concept of entropy of a fuzzy set (1968) can be interpreted as a Shannon entropy in which each term will be weighted by the corresponding fuzzy membership.

Garmendia et al. (2003) gives a general procedure to define *specificity measures,* departing of t-norms and negations.

The specificity measure is not necessarily a monotonic measure, because when a membership degree (which is not the greatest) is increasing, the specificity can to decrease.

Let A be a continuous domain.

Yager (1998) define a *Measure of Specificity on A.*

Previously (1991) define the *Specificity of a fuzzy set under similarities.*

Finally, we will analyze the *Specificity Measure,* as measure of degree of tranquillity when we take decisions.

Such *Specificity Measure* is a function with a new domain

$$Sp: \; [0,1]^U \to [0,1]$$

where $[0,1]^U$ is the class of fuzzy sets on the universe.

The *Sp function* must verifies

I) $Sp\ (\varnothing) = 0.$

II) $Sp\ (\delta) = 1$ if and only if δ *is a singleton*,

i.e. there exists some element $x \in U$ such that $\delta = \{x\}$.

III) If A and B are *normal fuzzy sets* in U, with $A \subset B$, then

$$Sp(A) \geq Sp(B)$$

Based on the aforementioned three axioms, *Yager* suggested, in 1982, this expression as a *Measure of Specificity of a finite and non-empty set,* A,

$$Sp(A) = \int_0^{h(A)} \frac{1}{card(A_\alpha)}\ d\alpha$$

being A_α the $\alpha - cut\ of\ A$, i. e.

$$A_\alpha = \{x : \mu_A(x) \geq \alpha\}$$

and $h(A)$ is the height of A.

From this expression, in the particulat case of a normal fuzzy set, i. e. when

$$h(A) = 1$$

it will be reduced to

$$Sp(A) = \int_0^1 \frac{1}{card(A_\alpha)}\ d\alpha$$

3.Conclusion

With this new approximation to fuzzy measures and their classification, we hope to contribute in the advance through the field of Uncertainty Measures, for to give an example of application. And so, advancing through the Approximate Reasoning. It will be also useful to work in different fields, such as Fuzzy Inference, Fuzzy Optimization, Fuzzy Analysis, Fuzzy and Rough Set Theory, and so on.

References

[1] Belis and Guiasu, *A qualitative-quantitative measure of information in cybernetics systems*, IEEE Trans. Info Theory, Volume IT-14, (1968), pages 593-594.

[2] De Luca and Termini, *Entropy of L-fuzzy sets.* Info and Control, Vol. 24, (1974), pp. 55-73.

[3] Dubois and Prade, *A note on measures of fuzziness and fuzzy complements.* Int. J. Gen. Systems, Vol.8, (1985), pp. 279-283.

[4] Higashi and Klir, *Measures of uncertainty and information based on possibility distributions.* Int. J. Gen. Syst., vol. 9 (1983), pp. 43-58.

[5] Jumaire, *Subjectivity, information, systems - introduction to a theory of relatvistic cybernetics.* Gordon and Breach Science Publishers. New York, 1986.

[6] Kaufmann, *Introduction to the theory of fuzzy subsets.* Vol. 1. Academic Press, New York, 1975.

[7] Pal (N. R.), and Pal (S. K.), *Higher order fuzzy entropy and hybrid entropy of a set.* Info. Sc., vol. 61, issue 3 (1992), pp. 211-231.

[8] Sander, *On measures of fuzziness.* FSS, vol. 29, (1989), pp. 49-55.

[9] Trillas and Riera, *Entropies in finite fuzzy sets.* Info. Sciences. Vol. 15, (1987), pp. 159-168.

[10] Yager, *Measuring tranquility and anxiety in decision making: an application of fuzzy sets.* Int. J. Gen. Systems. Vol. 8. Issue 3, (1982), pp. 139-144.

[11] Yager, *Entropy and Specificity in a mathematical theory of evidence.* Int. J. Gen. Systems. Vol. 9 (1983), pp. 249-260.

[12] Xie and Bedrosian, *An information measure for fuzzy sets.* IEEE Trans. Syst. Man and Cybern., SMC-14, issue 1 (1984), pp. 151-156.

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