PROPERTIES OF SOME FAMILIES OF MEROMORPHIC P-VALENT FUNCTIONS INVOLVING CERTAIN DIFFERENTIAL OPERATOR

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Abstract. Making use of a differential operator, which is defined here by means of the Hadamard product (or convolution), we introduce the class $\Sigma_p^n(f,g;\lambda,\beta)$ of meromorphically p-valent functions. The main object of this paper is to investigate various important properties and characteristics for this class. Also a property preserving integrals is considered.

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1. Introduction

Let Σ_p denote the class of functions of the form:

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \quad (p \in N = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disc $U^* = \{z : z \in C$ and $0 < |z| < 1\} = U \backslash \{0\}$. For functions $f(z) \in \Sigma_p$ given by (1.1) and $g(z) \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{k=0}^{\infty} b_k z^k \quad (p \in N), \tag{1.2}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f * g)(z) = z^{-p} + \sum_{k=0}^{\infty} a_k b_k z^k = (g * f)(z). \tag{1.3}$$

For complex parameters $\alpha_1, \alpha_2, ..., \alpha_q$ and $\beta_1, \beta_2, ..., \beta_s$ ($\beta_j \notin Z_0^- = \{0, -1, -2, ...\}$; $j = 1, 2, ..., s$), we now define the generalized hypergeometric function ${}_qF_s\ (\alpha_1, \alpha_2, ..., \alpha_q; \beta_1, \beta_2, ..., \beta_s;\ z)$ by (see, for example, [10] and [11])

$$ {}_qF_s(\alpha_1,\alpha_2,...,\alpha_q;\ \beta_1,\beta_2,...,\beta_s;\ z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k...(\alpha_q)_k}{(\beta_1)_k...(\beta_s)_k(1)_k}z^k $$

$$ (q\le s+1;\ s,q\in N_0=N\cup\{0\};\ z\in U), \tag{1.4} $$

where $(\theta)_k$, is the Pochhammer symbol defined in terms of the Gamma function Γ, by

$$ (\theta)_v=\frac{\Gamma(\theta+v)}{\Gamma(\theta)}=\begin{cases}1 & (v=0;\ \theta\in C^*=C\backslash\{0\}),\\ \theta(\theta+1)....(\theta+v-1) & (v\in N;\ \theta\in C).\end{cases} $$

Corresponding to the function $h_p(\alpha_1,\alpha_2,...,\alpha_q;\ \beta_1,\beta_2,...,\beta_s;\ z)$, defined by

$$ h_p(\alpha_1,\alpha_2,...,\alpha_q;\ \beta_1,\beta_2,...,\beta_s;\ z)=z^{-p}\ {}_qF_s\ (\alpha_1,\alpha_2,...,\alpha_q;\ \beta_1,\beta_2,...,\beta_s;\ z), \tag{1.5} $$

we consider a linear operator

$$ H_p(\alpha_1,\alpha_2,...,\alpha_q;\ \beta_1,\beta_2,...,\beta_s):\Sigma_p\longrightarrow\Sigma_p, $$

which is defined by the following Hadamard product:

$$ H_p(\alpha_1,\alpha_2,...,\alpha_q;\ \beta_1,\beta_2,...,\beta_s)f(z)=h_p(\alpha_1,\alpha_2,...,\alpha_q;\ \beta_1,\beta_2,...,\beta_s;\ z)*f(z) $$

$$ (q\le s+1;\ s,q\in N_0;z\in U). \tag{1.6} $$

We observe that, for a function $f(z)$ of the form (1.1), we have

$$ H_p(\alpha_1,\alpha_2,...,\alpha_q;\ \beta_1,\beta_2,...,\beta_s)f(z)=H_{p,q,s}(\alpha_1)=z^{-p}+\sum_{k=0}^{\infty}\Gamma_k a_k z^k, \tag{1.7} $$

where

$$ \Gamma_k=\frac{(\alpha_1)_{k+p}...(\alpha_q)_{k+p}}{(\beta_1)_{k+p}...(\beta_s)_{k+p}(1)_{k+p}}. \tag{1.8} $$

Then one can easily verify from (1.7) that

$$ z(H_{p,q,s}(\alpha_1)f(z))'=\alpha_1 H_{p,q,s}(\alpha_1+1)f(z)-(\alpha_1+p)H_{p,q,s}(\alpha_1)f(z). \tag{1.9} $$

The linear operator $H_{p,q,s}(\alpha_1)$ was investigated recently by Liu and Srivastava [9] and Aouf [2]. The operator $H_{p,q,s}(\alpha_1)$ contains the operator $\ell_p(a,c)$ (see [8]) for $q=2,\ s=1,\ \alpha_1=a>0,\ \beta_1=c\ (c\neq 0,-1,...)$ and $\alpha_2=1$ and also contains the operator $D^{\nu+p-1}$ (see [1] and [4]) for $q=2,\ s=1,\ \alpha_1=\nu+p\ (\nu>-p;\ p\in N)$ and $\alpha_2=\beta_1=p$.

For functions $f,\ g\ \in \Sigma_p$, we define the linear operator $D^n_{\lambda,p}(f*g)(z) : \Sigma_p \longrightarrow \Sigma_p$ $(\lambda \geq 0;\ p \in N;\ n \in N_0)$ by

$$D^0_{\lambda,p}(f*g)(z) = (f*g)(z), \tag{1.10}$$

$$D^1_{\lambda,p}(f*g)(z) = D_{\lambda,p}(f*g)(z) = (1-\lambda)(f*g)(z) + \lambda z^{-p}\ (z^{p+1}(f*g)(z))',$$

$$= z^{-p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]a_k b_k z^k\ (\lambda \geq 0;\ p \in N), \tag{1.11}$$

$$D^2_{\lambda,p}(f*g)(z) = D_{\lambda,p}(D_{\lambda,p}(f*g))(z),$$

$$= (1-\lambda)D_{\lambda,p}(f*g)(z) + \lambda z^{-p}\ (z^{p+1}D_{\lambda,p}(f*g)(z))'$$

$$= z^{-p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]^2 a_k b_k z^k\ (\lambda \geq 0;\ p \in N) \tag{1.12}$$

and (in general)

$$D^n_{\lambda,p}(f*g)(z) = D_{\lambda,p}(D^{n-1}_{\lambda,p}(f*g)(z))$$

$$= z^{-p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]^n a_k b_k z^k\ (\lambda \geq 0;\ p \in N;\ n \in N_0). \tag{1.13}$$

From (1.13) it is easy to verify that:

$$z(D^n_{\lambda,p}(f*g)(z))' = \frac{1}{\lambda}D^{n+1}_{\lambda,p}(f*g)(z) - (p+\frac{1}{\lambda})D^n_{\lambda,p}(f*g)(z)\ (\lambda > 0). \tag{1.14}$$

In this paper, we introduce the class $\Sigma^n_p(f,g;\lambda,\beta)$ of the functions $f,\ g \in \Sigma_p$, which satisfy the condition:

$$\mathrm{Re}\left\{\frac{D^{n+1}_{\lambda,p}(f*g)(z)}{\lambda D^n_{\lambda,p}(f*g)(z)} - (p+\frac{1}{\lambda})\right\} < -\beta \quad (z \in U^*;\ \lambda > 0;\ 0 \leq \beta < p;\ p \in N;\ n \in N_0). \tag{1.15}$$

We note that:

(i) If $\lambda = 1$ and the coefficients $b_k = 1$ $\left(\text{or } g(z) = \dfrac{1}{z^p\,(1-z)}\right)$ in (1.15), the class $\Sigma^n_p(f,g;\lambda,\beta)$ reduces to the class $B_n(\beta)$ studied by Aouf and Hossen [3];

(ii) If $n = 0,\ \lambda = 1$ and $g(z) = \dfrac{1}{z^p\,(1-z)^{\nu+p}},\ (\nu > -p;\ p \in N)$ in (1.15), the class $\Sigma_p^n(f, g; \lambda, \beta)$ reduces to the class $M_{\nu+p-1}(\beta)$, studied by Aouf [1];
(iii) If $b_k = 1$ $\left(\text{or } g(z) = \dfrac{1}{z^p\,(1-z)}\right)$ in (1.15), the class $\Sigma_p^n(f, g; \lambda, \beta)$ reduces to the class $K_p^n(\lambda, \beta)$, where $K_p^n(\lambda, \beta)$ is defined by

$$\operatorname{Re}\left\{\frac{D_{\lambda,p}^{n+1}f(z)}{\lambda D_{\lambda,p}^{n}f(z)} - (p+\frac{1}{\lambda})\right\} < -\beta \qquad (z \in U^*;\ \lambda > 0;\ 0 \le \beta < p;\ p \in N;\ n \in N_0), \tag{1.16}$$

where

$$D_{\lambda,p}^{n}f(z) = z^{-p} + \sum_{k=0}^{\infty}[1+\lambda(k+p)]^n a_k z^k \ (\lambda > 0;\ p \in N;\ n \in N_0); \tag{1.17}$$

(iv) If $n = 0,\ \lambda = 1$ and the coefficients $b_k = \dfrac{(a)_k}{(c)_k}$ $(c \neq 0, -1, ...)$ in (1.15), the class $\Sigma_p^n(f, g; \lambda, \beta)$ reduces to $\Sigma_p^n(a, c; \lambda, \beta)$, where $\Sigma_p^n(a, c; \lambda, \beta)$ is defined by

$$\operatorname{Re}\left\{\frac{a\ \ell_p(a+1,c)f(z)}{\ell_p(a,c)f(z)} - (a+p)\right\} < -\beta \qquad (z \in U^*;\ 0 \le \beta < p;\ p \in N); \tag{1.18}$$

(v) If $n = 0,\ \lambda = 1$ and the coefficients b_k in (1.15) is replaced by Γ_k, where Γ_k is given by (1.8), the class $\Sigma_p^n(f, g; \lambda, \beta)$ reduces to $\Sigma_{p,q,s}^n(\alpha_1, \beta_1; \lambda, \beta)$, where $\Sigma_{p,q,s}^n(\alpha_1, \beta_1; \lambda, \beta)$ is defined by

$$\operatorname{Re}\left\{\frac{\alpha_1 H_{p,q,s}(\alpha_1+1)f(z)}{H_{p,q,s}(\alpha_1)f(z)} - (\alpha_1+p)\right\} < -\beta$$

$$(z \in U^*;\ \alpha_1 \in C^*;\ 0 \le \beta < p;\ p \in N). \tag{1.19}$$

In this paper known results of Bajpai [5], Goel and Sohi [6], Uralegaddi and Somanatha [12] and Aouf and Hossen [3] are extended.

2.Basic properties of the class $\Sigma_p^n(f, g; \lambda, \beta)$

We begin by recalling the following result (Jack's Lemma), which we shall apply in proving our first inclusion theorems (Theorem 1 and Theorem 2 below).

Lemma 1 *[7]. Let the (nonconstant) function $w(z)$ be analytic in U,with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then*

$$z_0 w^{'}(z_0) = \xi w(z_0)\,, \qquad (2.1)$$

where ξ is a real number and $\xi \geq 1$.

Theorem 1. *For* $\lambda > 0,\ 0 \leq \beta < p,\ p \in N$ and $n \in N_0$,

$$\Sigma_p^{n+1}(f, g; \lambda, \beta) \subset \Sigma_p^{n}(f, g; \lambda, \beta)\,. \qquad (2.2)$$

Proof. Let $f(z) \in \Sigma_p^{n+1}(f, g; \lambda, \beta)$. Then

$$\mathrm{Re}\left\{ \frac{D_{\lambda,p}^{n+2}(f * g)(z)}{\lambda D_{\lambda,p}^{n+1}(f * g)(z)} - (p + \frac{1}{\lambda}) \right\} < -\beta,\ |z| < 1. \qquad (2.3)$$

We have to show that (2.3) implies the inequality

$$\mathrm{Re}\left\{ \frac{D_{\lambda,p}^{n+1}(f * g)(z)}{\lambda D_{\lambda,p}^{n}(f * g)(z)} - (p + \frac{1}{\lambda}) \right\} < -\beta. \qquad (2.4)$$

Define a regular function $w(z)$ in U by

$$\frac{D_{\lambda,p}^{n+1}(f * g)(z)}{\lambda D_{\lambda,p}^{n}(f * g)(z)} - (p + \frac{1}{\lambda}) = -\frac{[p + (2\beta - p)w(z)]}{1 + w(z)}. \qquad (2.5)$$

Clearly $w(0) = 0$. Equation (2.5) may be written as

$$\frac{D_{\lambda,p}^{n+1}(f * g)(z)}{D_{\lambda,p}^{n}(f * g)(z)} = \frac{1 + [1 + 2\lambda(p - \beta)]w(z)}{1 + w(z)}. \qquad (2.6)$$

Differentiating (2.6) logarithmically with respect to z and using (1.14), we obtain

$$\frac{\frac{D_{\lambda,p}^{n+2}(f*g)(z)}{\lambda D_{\lambda,p}^{n+1}(f*g)(z)} - (p + \frac{1}{\lambda}) + \beta}{(p - \beta)} = \frac{2\lambda z w'(z)}{(1+w(z))\{1+[1+2\lambda(p-\beta)]w(z)\}} - \frac{1-w(z)}{1+w(z)}. \qquad (2.7)$$

We claim that $|w(z)| < 1$ for $z \in U$. Otherwise there exists a point $z_0 \in U$ such that $\max\limits_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1$. Applying Jack's Lemma, we have $z_0 w^{'}(z_0) = \xi w(z_0) (\xi \geq 1)$. Writing $w(z_0) = e^{i\theta} (0 \leq \theta \leq 2\pi)$ and putting $z = z_0$ in (2.7), we get

$$\frac{\frac{D_{\lambda,p}^{n+2}(f*g)(z_0)}{\lambda D_{\lambda,p}^{n+1}(f*g)(z_0)}-(p+\frac{1}{\lambda})+\beta}{p-\beta}=\frac{2\lambda\xi w(z_0)}{(1+w(z_0))\{1+[1+2\lambda(p-\beta)]w(z)\}}-\frac{1-w(z_0)}{1+w(z_0)}. \tag{2.8}$$

Thus

$$\operatorname{Re}\left\{\frac{\frac{D_{\lambda,p}^{n+2}(f*g)(z_0)}{\lambda D_{\lambda,p}^{n+1}(f*g)(z_0)}-(p+\frac{1}{\lambda})+\beta}{p-\beta}\right\}\geq\frac{1}{2[1+\lambda\,(p-\beta)]}>0, \tag{2.9}$$

which obviously contradicts our hypothesis that $f(z)\in\Sigma_p^{n+1}(f,g;\lambda,\beta)$. Thus we must have $|w(z)|<1$ $(z\in U)$, and so from (2.5), we conclude that $f(z)\in\Sigma_p^{n}(f,g;\lambda,\beta)$, which evidently completes the proof of Theorem 1.

Theorem 2. *Let $f,g\in\Sigma_p$ satisfy the condition*

$$\operatorname{Re}\left\{\frac{D_{\lambda,p}^{n+1}(f*g)(z)}{\lambda D_{\lambda,p}^{n}(f*g)(z)}-(p+\frac{1}{\lambda})\right\}<-\beta+\frac{(p-\beta)}{2(p-\beta+c)}\quad(z\in U) \tag{2.10}$$

for $\lambda>0,\ 0\leq\beta<p,\ p\in N,\ n\in N_0$ *and* $c>0$. *Then*

$$F_{c,p}(f*g)(z)=\frac{c}{z^{c+p}}\int_0^z t^{c+p-1}(f*g)(t)dt \tag{2.11}$$

belongs to $\Sigma_p^n(f,g;\lambda,\beta)$.

Proof. From the definition of $F_{c,p}(f*g)(z)$, we have

$$z(D_{\lambda,p}^{n}F_{c,p}(f*g)(z))'=cD_{\lambda,p}^{n}(f*g)(z)-(c+p)D_{\lambda,p}^{n}F_{c,p}(f*g)(z) \tag{2.12}$$

and also

$$z(D_{\lambda,p}^{n}F_{c,p}(f*g)(z))'=\frac{1}{\lambda}D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)-(p+\frac{1}{\lambda})D_{\lambda,p}^{n}F_{c,p}(f*g)(z)\ (\lambda>0). \tag{2.13}$$

Using (2.12) and (2.13), the condition (2.10) may be written as

$$\operatorname{Re}\left\{\frac{\frac{D_{\lambda,p}^{n+2}F_{c,p}(f*g)(z)}{\lambda D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)}+(c-\frac{1}{\lambda})}{1+(\lambda c-1)\frac{D_{\lambda,p}^{n}F_{c,p}(f*g)(z)}{D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)}}-(p+\frac{1}{\lambda})\right\}<-\beta+\frac{p-\beta}{2(p-\beta+c)}. \tag{2.14}$$

We have to prove that (2.14) implies the inequality

$$\operatorname{Re}\left\{\frac{D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)}{\lambda D_{\lambda,p}^{n}F_{c,p}(f*g)(z)}-(p+\frac{1}{\lambda})\right\}<-\beta. \tag{2.15}$$

Define a regular function $w(z)$ in U by

$$\frac{D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)}{\lambda D_{\lambda,p}^{n}F_{c,p}(f*g)(z)}-(p+\frac{1}{\lambda})=-\frac{[p+(2\beta-p)w(z)]}{1+w(z)}. \tag{2.16}$$

Clearly $w(0)=0$. The equation (2.16) may be written as

$$\frac{D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)}{D_{\lambda,p}^{n}F_{c,p}(f*g)(z)}=\frac{1+[1+2\lambda(p-\beta)]w(z)}{1+w(z)}. \tag{2.17}$$

Differentiating (2.17) logarithmically with respect to z and using (2.13), we obtain

$$\frac{D_{\lambda,p}^{n+2}F_{c,p}(f*g)(z)}{\lambda D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)}-\frac{D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)}{\lambda D_{\lambda,p}^{n}F_{c,p}(f*g)(z)}=\frac{2\lambda(p-\beta)zw'(z)}{(1+w(z))[1+(1+2\lambda(p-\beta))w(z)]}. \tag{2.18}$$

The above equation may be written as

$$\frac{\frac{D_{\lambda,p}^{n+2}F_{c,p}(f*g)(z)}{\lambda D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)}+(c-\frac{1}{\lambda})}{1+(\lambda c-1)\frac{D_{\lambda,p}^{n}F_{c,p}(f*g)(z)}{D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)}}-(p+\frac{1}{\lambda})=\frac{D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)}{\lambda D_{\lambda,p}^{n}F_{c,p}(f*g)(z)}-(p+\frac{1}{\lambda})$$

$$+\left[\frac{2\lambda(p-\beta)zw'(z)}{(1+w(z))[1+(1+2\lambda(p-\beta))w(z)]}\right]\left[\frac{1}{1+(\lambda c-1)\frac{D_{\lambda,p}^{n}F_{c,p}(f*g)(z)}{D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)}}\right],$$

which, by using (2.16) and (2.17), reduces to

$$\frac{\frac{D_{\lambda,p}^{n+2}F_{c,p}(f*g)(z)}{\lambda D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)}+(c-\frac{1}{\lambda})}{1+(\lambda c-1)\frac{D_{\lambda,p}^{n}F_{c,p}(f*g)(z)}{D_{\lambda,p}^{n+1}F_{c,p}(f*g)(z)}}-(p+\frac{1}{\lambda})=-\left[\beta+(p-\beta)\frac{1-w(z)}{1+w(z)}\right]$$

$$+\frac{2\lambda(p-\beta)zw'(z)}{(1+w(z))\{c+[c+2(p-\beta)]w(z)\}}. \tag{2.19}$$

The remaining part of the proof is similar to that Theorem 1, so we omit it.

Remark 1. *(i) For* $\lambda = p = c = a_k = b_k = 1$ *and* $n = \beta = 0$, *we note that Theorem 2 extends a results of Bajpai [5, Theorem 1];*

(ii) For $\lambda = p = a_k = b_k = 1$ *and* $n = \beta = 0$, *we note that Theorem 2 extends a results of Goel and Sohi [6, Corollary 1].*

Theorem 3. $(f * g)(z) \in \Sigma_p^n(f, g; \lambda, \beta)$ *if and only if*

$$F(f * g)(z) = \frac{1}{z^{1+p}} \int_0^z t^p (f * g)(t) dt \in \Sigma_p^{n+1}(f, g; \lambda, \beta). \tag{2.20}$$

Proof. From the definition of $F(f * g)(z)$ we have

$$D_{\lambda,p}^n (zF'(f * g)(z)) + (1 + p) D_{\lambda,p}^n F(f * g)(z) = D_{\lambda,p}^n (f * g)(z),$$

that is,

$$z(D_{\lambda,p}^n F(f * g)(z))' + (1 + p) D_{\lambda,p}^n F(f * g)(z) = D_{\lambda,p}^n (f * g)(z). \tag{2.21}$$

By using the identity (1.14), (2.21) reduces to $D_{\lambda,p}^n (f*g)(z) = D_{\lambda,p}^{n+1} F(f*g)(z)$. Hence $D_{\lambda,p}^{n+1} (f * g)(z) = D_{\lambda,p}^{n+2} F(f * g)(z)$, therefore,

$$\frac{D_{\lambda,p}^{n+1} (f * g)(z)}{D_{\lambda,p}^n (f * g)(z)} = \frac{D_{\lambda,p}^{n+2} F(f * g)(z)}{D_{\lambda,p}^{n+1} F(f * g)(z)}$$

and the result follows.

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