EQUIVALENT MARTINGALE MEASURES FOR LÉVY PROCESSES

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Abstract. In this paper, we study the equivalent martingale measures in Lévy market models. One of the important questions in financial modelling is the choice of an appropriate model for financial assets. Lévy processes have proved to be particularly suited for modelling market price fluctations. The existence and uniqueness of martingale measures is an important problem in financial modelling. Because, this problem is related to the absence of arbitrage and completeness of the corresponding models of price evolution. Lévy models include jumps. Jumps are useful to capture unexpected changes in the market that other diffusion processes are not capable to sketch. Since, the inclusion of jump Lévy processes in the model leads to incomplete markets. In an incomplete market, the set of the martingale measure is infinite and some also be the set of prices for a single derivative. In this case, one needs to specify in some way one martingale measure in order to get a pricing rule and uniqueness of prices for derivatives.

Keywords and Phrases: Lévy Processes, Equivalent Martingale Measures, Mathematical Finance, Moment Estimation.

1. Introduction

Levy processes (Processes with Stationary Independent Increments)are popular mathematical tools in the Physics, Engineering, Mathematical finance ext. ([12], [14]).Infinitely divisible distributions and Lévy processes have been the subject of intense research and applications in recent years. Because, their paths can be decomposition into a Brownian motion with drift plus an independent superposition of jumps of all possible size. This decomposition of Lévy processes makes them suited for modelling random pheonomena which manifest jumps.([1],[2],[6],[20],[18],[4]).Lévy processes has provided to be a good mathematical method, especially for models of the stock markets. It is a well – known fact that returns from financial market variables which measured daily or weekly are characterized by non normality([16]).

The emprical distribution of such returns has fat tails, asymmetry and autocorrelation([17]).These stylized facts are easily described by models based on Lévy processes. Lévy processes have some important features such that these processes have paths consist of continuous motion interspersed with jump discontinuities of random size. They are natural model of noise that can be used to build stochastic integrals and to drive stochastic differential eqations. Their structure is mathematical robust([2]). In incomplete market, derivative prices are not determined by no arbitrage. The prices depends on investors preferences. One approach to find the correct equivalent martingale measure consist in trying to identify a utility function describing the investors preferences([10]).Another a possible choice for an equivalent martingale measure when the asset return is modelled by a Lévy processes is Esscher transform ([9]).

When pricing derivatives in a financial market, in generally ones use no arbitrage arguments. The price of derivative is given by an expectation of the discounted payoff of the derivative, the expectation taken with respec to some Q equivalent martingale measure. Q equivalent martingale measure can be obtain by using Cameron – Martin –Girsanov theorem. If the market incomplete the measure Q is not unique([13]).

The paper is organized as follows. In section 2, we provide a brief introduction to Lévy processes and we describe probabilistic structure and path properties of Lévy processes. In section 3 we introduce relationship between martingale and Lévy processes. In section 4 we introduce financial modelling with exponential Lévy models. In section 5, we have the conclusions.

2. Lévy processes

In this section we introduce definition and some basic properties of Lévy processes. The name Lévy processes honour the work of the French mathematician Paul Lévy(1886-1971). General references on Lévy processes are [18],[4],[3],[6], [20],[1].

Definition 2.1 (Lévy processes). *A cadlag, adapted, real valued stochastic process* $X = (X_t)_{t\geq 0}$ *defined on a filtred probability space* $\left(\Omega, \Im, (\Im_t)_{t\geq 0}, P\right)$ *is said to be a Lévy processes the following properties.*

(i) $P(X_0 = 0) = 1$

(ii) *For* $0 < s < t$ *,the distribution of* $X_t - X_s$ *is equal in distribution to* X_{t-s} *.i.e, X has stationary increments.*

(iii) *For* $0 < s < t$ *,the distribution of* $X_t - X_s$ *is dependent of* $\{X_u; u \leq s\}$ *.i.e, X has independend increments.*

(iv) *All* $a > 0$ *and for all* $s \geq 0$, $\lim_{s\to t} P(|X_t - X_s| > a) = 0$. *i.e, X is stochastically continuous.*

Lévy processes has a vesion with cadlag paths, i.e. paths which are right continuous and have limits from the left ([15],theorem 30.). We shall always consider processes with cadlag paths. Note that the (iv) condition does not imply that the path of Lévy processes are continuous. It only requires that for a given time t ,the probability of seeing a jump at t is zero, i.e. jumps occur at random times.

Definition 2.2 (Characteristic Function). *The charecteristic function* ϕ *of a random variable X is the Furier-Stieltjes transform of the distribution function* $F(x) = P(X \leq x)$ *:*

$$\phi_X(u) = E[\exp(iuX)] = \int_{-\infty}^{\infty} \exp(iuX)\, dF(x)$$

If random variable X has acontiuous distribution with density function $f_X(x)$ above equation become,

$$\phi_X(u) = E[\exp(iuX)] = \int_{-\infty}^{\infty} \exp(iuX)\, f_X(x)\, dx \qquad (1)$$

Some functions related to the charecteristic function are following([5],[20], [14]).

- The cumulant function: $\kappa(u) = \log E[\exp(-uX)] = \log \phi(iu)$
- The moment generating function: $\varphi(u) = E[\exp(uX)] = \phi(-iu)$
- The characteristic exponent (The cumulant characteristic function):

$$\psi(u) = \log E[\exp(iuX)] = \log \phi(u) \text{ or equivalently } \phi(u) = \exp[\psi(u)] \, .$$

There is one to one correspondance between infinitelly divisible distributions and Lévy processes.

Difinition 2.3 (Infinitely Divisible Distributions). *A random variable X is infinity divisible, if for all $n \in N$,there exist i.i.d. random variables $X_1^{(1/n)}, ..., X_n^{(1/n)}$ such that*

$$X =^d X_1^{(1/n)} + ... + X_n^{(1/n)} \tag{2}$$

Definition 2.4. *The law of a random variable X is infinitely divisible if any integer $n > 0$,there exist a probability measure μ_n such that its characteristic function $(\phi_\mu(u)) = E_\mu[\exp(iuX)]$ satisfies*

$$\phi_\mu(u) = (\phi_{\mu_n}(u))^{\,n} \tag{3}$$

2.1 Probabilistic Structure and path properties of Lévy Processes

The characteristic functions of infinitely divisible probability measures were completely characterized by Lévy and Khintchine in 1930's.The characteristic function of a Lévy processes is given by Lévy-Khintchine Formula,

Theorem 2.1(Lévy-Khintchine Formula). *If $X = (X_t)_{t\geq 0}$ is a Lévy processes, then $\phi_t(u) = e^{t\,\psi(u)}$ for each $t \geq 0$, $u \in \Re$, where*

$$\psi(u) = iub - \frac{1}{2}u^2 c + \int_{\Re} \left(e^{iux} - 1 - iux_{\{|x|<1\}}\right)\, v(dx) \tag{4}$$

is called the Characteristic exponent. The (b, c, v) is called the Lévy –Khintchine triplet or charecteristic triplet. In here $b \in R$ is called drift term, $c \in R_+$ diffusion coefficient and v the Lévy measure. v is a positive measure on R describing the jumps of the processes([2]). If X is compound Poisson, then $v(R) < \infty$ and $v(dx) = \lambda f(dx)$ but in the general case v need not to be finite measure. It must satisfy the constraints: v does not have mass on 0 , $v(\{0\}) = 0$ and $\int_R (1 \wedge x^2)\, v(dx) < \infty$ v describes the jumps of X processes in the following sense: for every closed set $A \subset R$ with $0 \notin A$, $v(A)$ is the average number of jumps of X in the time interval $[0, 1]$, whose sizes fall in A ([6]). $h(x) = x1_{\{|x|<1\}}$ is the truncation function. Changing h changes drift parameter b, but diffusion coefficient $c \in R_+$ and v Lévy measure remain uneffected. The distribution of Lévy process $X = (X_t)_{t\geq 0}$ is completely determined by any of its marginal distributions $\ell(X_t)$. From definition of Lévy processes, we see that for any $t > 0$, X_t is a random variable belonging to the class of infinitely divisible distributions. For any natural n and $t > 0$,

$$X_t = X_{t/n} + \left(X_{2t/n} - X_{t/n}\right) + ... + \left(X_t - X_{(n-1)t/n}\right) \tag{5}$$

Together with stationary and independence of the increments, random variable X_t is infinitely divisible. For any natural m , $\psi_m(u) = m\psi_1(u)$, Hence, for any $t > 0$, $\psi_t(u) = t\psi_1(u)$ or $E[\exp(iuX_1)] = \exp(\psi(u))$. For any $n \in N$, we can denote

$$X_1 = \sum_{j=1}^{n} \left(X_{j/n} - X_{(j-1)/n}\right) \tag{5-a}$$

By stationary and independence of the increments,

$$\ell(X_1) = \ell\left(X_{1/n}\right) * ... * \ell\left(X_{1/n}\right) \tag{6}$$

As a result, the Lévy process is determined as soon as we know X_1. This property is useful when one has to compute numerically values of derivatives which are represented as expectations of the form $E[f(X_T)]$ where X_T is the value of a Lévy processes at maturity T, and the parameters of the Lévy processes were estimated as the parameters of $\ell(X_1)$ ([7]).

2.2 The Lévy-Ito Decomposition

In this section, we try to understand the structure of Lévy processes. Given a characteristic exponent, we can always associate to it a Lévy processes with cadlag path. Such a process has at most finite jump discontinuous on each closed interval.We can denote one dimensional Lévy processes in the following way,

$$X_t = bt + \sqrt{c}\, W_t + \sum_{0 \le s \le t} \Delta X_s \tag{7}$$

Where $\Delta X_s = X_s - X_{s^-}$ is the jump at time s , where $X_{s^-} = \lim_{u \to s} X_u$

If $c = 0$, the process is called a pure continuous Lévy processes. $X_b = bt + W_t$ process is a Brownian motion with drift. Last term in equation (7) denotes jumps of X Lévy processes. This sum may not converge. Now we introduce a new measure to count the jumps([7],[6]).

$$J(t,A) = \#\left(\Delta X_s \in A\,;0 \leq s \leq t\right) \tag{8}$$

The jump measure of X denoted by J is called Poisson random measure with intensity measure $v\,(dx)\;dt$. Where, $A \in B\,(R)$ and

$$v(A) = E\left[\#\left(t \in [0,1]\;;\;\Delta X_t \neq 0, \Delta X \in A\right)\right] \tag{9}$$

X can have only a finite number of jumps of size greater than 1 , $\int\limits_{|x|>1} xJ(t,dx) < \infty$. So the sum, $\sum\limits_{0\leq s\leq t} \Delta X_s 1_{\{|\Delta X_s|\,>1\}}$ contains almost a finite number of terms. Contrary to the sum of the big jumps, the sum of the small jumps , $\sum\limits_{0\leq s\leq t} \Delta X_s 1_{\{|\Delta X_s|\,\leq 1\}}$ .Another way we can represent the sum of the all jumps of size greater than ϵ but less than 1.

$$\sum_{0\leq s\leq t} \Delta X_s 1_{\{\epsilon\leq|\Delta X_s|\,<1\}} = \int\limits_{\epsilon\leq|x|\leq 1} x\,J\,(t,dx) \tag{10}$$

But this sum does not coverge in general for $\epsilon \to 0$. There are too many jumps to get coverge. In order to obtain covergence we have to subtract average increas of the process along $[0,t]$ .So we consider following process , $X_t^\epsilon = \int\limits_{\epsilon\leq|x|<1} x\;(J\,(t,dx) - t\,v\,(dx))$. This is sequence of square integrable, mean zero martingale([2]). The following result is called Lévy-Ito Decomposition for Lévy processes.

$$X_t = bt + X_t^c + \int\limits_0^t \int\limits_{|x|\geq 1} x\,J\,(ds,dx) + \left(\int\limits_0^t \int\limits_{|x|<1} x\,J(ds,dx) - t\int\limits_{|x|<1} x\,v\,(dx)\right) \tag{11}$$

where $X_t^c = \sqrt{c}\,W_t$ and W is a standart Brownian motion.

If $\int_{|x|\leq 1} |x|\;v\,(dx)\;<\;\infty$, a Lévy process X is called finite variation. In this case, The Lévy- Ito decomposition of X is be follow,

$$X_t = \gamma\,t + \sqrt{c}\,W_t + \int\limits_0^t \int\limits_{-\infty}^{\infty} x\,J\,(ds,dx) \tag{12}$$

The **Lévy- Khintchine formula** takes the following form,

$$E\left[\exp(iuX_t)\right] = \exp\left[t\left(iu\gamma - \frac{c}{2}u^2 + \int\limits_{\Re}\left(e^{iux}-1\right)v\left(dx\right)\right)\right] \qquad (13)$$

where, $\gamma = b + \int\limits_0^1 xv\left(dx\right) \quad > 0$ if $\int_{|x|\geq 1}|x|\ v\left(dx\right) \ < \infty$. A Lévy process satisfies $E\left[|X_t|\right] \ < \infty$. In this case, The Lévy- Ito decomposition of X ,

$$X_t = b't + \sqrt{c}\,W_t + \left(\int\limits_0^t\int\limits_{\Re} x\,J(ds,dx) - t\int\limits_{\Re} x\,v\left(dx\right)\right) \qquad (14)$$

and Lévy- Khintchine formula takes the form

$$E\left[\exp(iuX_t)\right] = \exp\left[t\left(iub\prime - \frac{c}{2}u^2 + \int\limits_{\Re}\left(e^{iux}-1-iux\right)v\left(dx\right)\right)\right] \qquad (15)$$

where $b' = b + \int_{|x|\geq 1}\ xv\left(dx\right)$ ([14]),p.13).

Theorem 2.2 (Doleeans-Dade Exponential). *Let* $X = (X_t)_{t\geq 0}$ *be a Lévy process. The stochastic differential equation*

$$dZ = Z_{t^-}dX_t \qquad Z_0 = 1 \qquad (16)$$

has a unique solution,

$$Z_t = \Xi\left(X_t\right) = e^{X_t - \frac{c}{2}t}\prod_{s\leq t}\left(1+\Delta X_s\right)\ e^{-\Delta X_s} \qquad (17)$$

where $\Delta X_t = X_t - X_{t^-}$ ([11]).

3.Equivalent Martingale measures

A Equivalent Martingale Measure(EMM) is an absolutely continuous probability measure respect to objective measure P .The set of EMM is not empty, morever it can have more than one EMM. The measure can be change with a deterministic process β and a non negative deterministic process Y i.e. β and Y are deterministic quantities. h function is called truncation function and $h : R \to R$ is given by $h(x) = x1_{0\leq|x|\leq 1}$.

Theorem 3.1 *Let P be a probability measure. Let $X = (X_t)_{t\geq 0}$ be a Lévy process with triplet $(b, c, v)_P$ .Then there is a probability measure $Q \sim P$ such that X is a Q -Lévy process with triplet $(b', c', v')_Q$ if and only if there exist $\beta \in R$ and a $y(x)$ function satisfying*

$$\int\limits_R |h(x)(1 - y(x))| \, v(dx) \quad < \infty \text{and} \int_R \left(1 - \sqrt{y(x)}\right)^2 v(dx) \quad < \infty$$

and

$$b' = b + c\beta + \int_R h(x)\,[y(x) - 1]\, v(dx)$$

$$c' = c$$

$$\frac{dv'}{dv} = y(x)\,.$$

For the proof of theorem([11]).

Remark. The β changes the drift of diffusion part of Lévy process X . The $y(x)$ function describes the jump structure under new measure. For every $A \in B(R)$ this function changes the jump density from $v(A)$ to $\int_A y(x)\, v(dx)$.

Martingale density process with β and y has following form,

$$Z_t = \left.\frac{dQ}{dP}\right|_{\Im_t} = \Xi\,(X_.)_{0\leq t\leq T} \tag{18}$$

$$X = \beta W + (y(x) - 1)(J_X - v) \tag{19}$$

Proposition 3.1 *If $\int_R \left(1 - \sqrt{y(x)}\right)^2 v(dx) \quad < \infty$ then above Z is a uniform integrable P -martingale.*

Theorem 3.2 *We suppose $\theta \in R$, $T > 0$ and $E[\exp(\theta X_1)] \ < \infty$, then defines a probability measure Q such that $Q \sim P$ and $X = (X_t)_{0\leq t\leq T}$ is a Lévy processes under Q with triplet $\left(b^\theta, c^\theta, v^\theta\right)$ given by*

$b^{\theta} = b + c\theta + \int \left(e^{\theta x} - 1\right) h(x)\, v(dx)$
$c^{\theta} = c$
$v^{\theta}(dx) = e^{\theta x} v(dx)$
For proof see ([21],p685).

$$E\left[e^{uX_t}\right] = e^{t\,\psi^{\theta}(u)} \tag{20}$$

where $\psi^{\theta}(u) = \psi(u+\theta) - \psi(\theta)$

Any martingale measure under Q follows under equation,

$$E_Q\left(\exp\left(X_t\right)\right) = \exp\left(rt\right) = \exp\left(t\,\psi^{\theta}(1)\right) \tag{21}$$

The parameter θ is obtained solving above equation. The measure Q is called Esscher transform of P .

4. Financial modelling

The correct modelling of asset returns is very important for derivative pricing and financial risk management. The classic model for stock prices and Indices is geometric Brownian motion. This model is given by following stochastic differential equation

$$dS_t = \mu\, S_t + \sigma\, S_t dW_t \tag{22}$$

This equation is solved by

$$S_t = S_0 \exp\left\{\ \left(\mu - \sigma^2/2\right)\ t + \sigma\, W_t\right\} \tag{23}$$

The constant $\mu \in R$ is the expected rate. $\sigma > 0$ called the volatility is a measure of the excitability of market. This price processes model have no any jumps.

4.1 Exponential Lévy Model

The emprical studies of stock prices have found evidance of heavy tails which is incompatible with a Gaussian model. This suggest that it might be fruitful to replace Brownian motion with more general Lévy processes.

Asset returns are usually defined as increments of log asset prices such that $X_t = \log S_t - \log S_{t-1}$ .We think of $X = (X_t)_{t\geq 0}$ as Lévy process. $S_0 > 0$ and we model the asset price process $(S_t)_{t\geq 0}$ with a exponential Lévy process that is

$$S_t = S_0 \exp(X_t) \tag{24}$$

By using Ito Formula, one can obtain stochastic differential equation

$$dS_t = S_{t^-} dX_t \tag{25}$$

The unique solution of stochastic differential equation,

$$S_t = S_0 \exp(X_t - ct/2) \prod_{s \le t} (1 + \Delta X_s) \exp(-\Delta X_s) \tag{26}$$

$\Delta X_t = X_t - X_{t^-}$ is the jump at time t.To ensure that $S_t > 0$ we need that $\Delta X_t > -1 \quad , t \in [0, T]$, In principle must describe equation (24) as following([7],p.8).

$$dS_t = S_{t^-} \left[dX_t + (c/2)\, dt + e^{\Delta X_t} - 1 - \Delta X_t \right] \tag{27}$$

In the context of derivative pricing one must choice parameter θ such that discount asset prices becomes a martingale under Q^θ. Because pricing is done by taking expectations under a martingale measure. For $(S_t)_{0 \le t \le T}$ to be a martingale

i) $E[S_t] < \infty$

ii) $E[\exp(X_t)] < \infty$

$S_t = S_0 \exp(X_t)$ process is a martingale if

$$b + \frac{c}{2} + \int_R (e^x - 1 - x)\ v(dx) = 0 \tag{28}$$

An exponential Lévy model is arbitraj free if and only if the trajectories of X are not almost surely increasing nor almost surely decreasing([22],s.7).To satisfy the martingale constraint in equation(27) we must change the Lévy mesure i.e. the intensity of jumps. In a pure diffusion case, for example, geometric Brownian motion EMM has following form,

$$Z_t = \frac{dQ}{dP} = \exp\left[\left(\frac{r - \mu}{\sigma} \right) W_t - \frac{(\mu - r)^2}{2\sigma^2}\, t \right] \tag{29}$$

But the jumps in price model will effect density of EMM. Under the unique EMM Q , the drift changes to the risk free interest rate r and the solution of the stochastic differential equation under Q is then obtained as follows,

$$S_t = S_{t-1} \exp\left(r - \sigma^2/2 + \sigma\left(W_t^Q - W_{t-1}^Q\right)\right) \tag{30}$$

In the case of Poisson process with drift, $X_t = N_t - bt \quad , b > 0$, we can find a martingale process by changing the intensity of N_t Poisson process to $\lambda = b/\left(e-1\right)$ ([22]).

The exponential Lévy model is a popular alternative to the geometric Brownian motion for financial modelling. Because a Lévy process is uniquely defined only one marginal distribution. In the contingent claim valuation, we find a unique equivalent martingale measure. In incomplete market case there is a class of such measures. A Lévy process gives rise a incomlete market. In incomplete market, option can not be replication. There exist a lot of sub and super hedging strategies.

$$P_{\sup} = \inf\left\{P \mid P \;\; is\; price\; of\; a\; \sup erhedge\right\},$$

$$P_{sub} = \sup\left\{P \mid P \;\; is\; price\; of\; a\; \sup erhedge\right\},$$

$$P \in \left(P_{sub}, P_{\sup}\right). \tag{31}$$

Now we must obtain a Q probability measure which has properties following,

i) Q is equivalent P,

ii) $e^{-r\,t} S_t$ is a martingale

There are many such $Q's$ .We can choice one of them using the Esscher transform. Esscher transform is structure preserving. Let X_t is a Lévy process under Q ,Radon-Nikodym derivative,

$$\frac{dQ}{dP} = e^{(\theta\, X_T - \,\phi(\theta)\; T)} \tag{32}$$

ϕ is the log-moment generating function of X_1 . θ chosen so that $e^{-r\,T} S_T$ has expectation S_0 .We can represent this as follows

$$\begin{aligned} S_0 &= E^Q\left[e^{-r\,T} S_T\right] \\ &= e^{-r\,T} S_0\; E\;\left[e^{X_T} e^{\theta\, X_T}\right]\; e^{-\phi(\theta)\,T} \quad (32) \\ &= e^{r\,T} S_0\; e^{\phi(1+\theta) - \phi(\theta)\,T} \end{aligned}$$

We choise θ so that

$$r = \phi\left(1+\theta\right) - \phi\left(\theta\right). \tag{33}$$

If compute values on equivalent martingale measures, we must span a bid-ask interval. For a European call option with strike price K and maturity T and with constant interest rate r, interval $\left\{\left(S_0 - \exp\left(-rT\right)K, 0\right)^{+}, S_0\right\}$ is include all price whatever model is used for asset prices. If we consider arbitrage argument than the range of the values of the options is $\left(\exp\left(-rT\right)H\left(\exp\left(rT\right)S_0\right), S_0\right)$ ([8],[22]).

5. Conclusion

The law of a Lévy process X_t is completely determined by the law of X_1. The class of Lévy models is include the Brownian model as a special case but contras to the brownian model, allow us to modelling jumps, skewness and access kurtosis in the price paths. General Lévy models are appropriate tools for asset price. But the Lévy process models leads incomplete market models. In these models only arbitrage free argument is not enough for derivative valuation. There is a range of arbitrage free prices. In this point to choice a particular equivalent martingale measure is needed some additional assumptions and suplementary information. To choice a risk neutral measure Q .Esscher transform is a appropriate tool. In the Esscher transform θ is chosen such that the discounted asset price $\tilde{S}_t$ is a martingale under new equivalent martingale measure. Lévy processes is sometimes insuffucient for multiperiod financial modelling. Because on a fixed time horizon, Lévy process have same law. In this case we can not include new information to model.

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