CHARACTERIZATIONS OF NONNEGATIVE SELFADJOINT EXTENSIONS

Adrian Sandovici

Abstract. The set of all nonnegative selfadjoint extensions of a nonnegative linear relation (multi-valued operator) is described by using a partial order defined on the set of the corresponding quadratic forms. This characterization leads to a generalization of a result due to M.G. Kreĭn.

2000 *Mathematics Subject Classification*: 47A06, 47A20, 47B25, 47A63.

1. Introduction

This paper deals with the problem of the extension theory of nonnegative linear relations (nonnegative multi-valued operators) in Hilbert spaces (see [1] for the case of closed nonnegative operators and [15] for the case of bounded nonnegative operators). The main result of this note - namely Theorem 1 - extends the applicability of a characterization of all nonnegative selfadjoint operator extensions of a nonnegative operator (proved in [14]) to the case of nonnegative linear relations (see also [2,3,6,8,9,10,12,13 for related papers). This characterization is also translated into the language of quadratic forms.

The paper is organized as follows. Section 2 contains a short introduction to linear relations in Hilbert spaces. Closed nonnegative forms are discussed in Section 3, and nonnegative selfadjoint extensions of nonnegative linear relations are presented in Section 4. Inequalities for nonnegative linear relations are introduced in Section 5, and results concerning the class of all nonnegative selfadjoint extensions of a nonnegative linear relation are obtained in Section 6. Finally, a Kreĭn's criteria is extended to the case of nonnegative linear relations on Section 7. A complete version of this note is available in [11].

2. Linear relations in Hilbert spaces

Let $\mathfrak{H}$ be a complex Hilbert space. A linear subspace A in the Cartesian product $\mathfrak{H} \times \mathfrak{H}$ is called a linear relation in $\mathfrak{H}$. Its domain, range, kernel, and multi-valued part are denoted by $\operatorname{dom} A$, $\operatorname{ran} A$, $\ker A$, and $\operatorname{mul} A$:

$$\begin{aligned} \operatorname{dom} A &= \{\, f \in \mathfrak{H} : \{f, f'\} \in A \,\text{for some}\, f' \in \mathfrak{H} \,\}, \\ \operatorname{ran} A &= \{\, f' \in \mathfrak{H} : \{f, f'\} \in A \,\text{for some}\, f \in \mathfrak{H} \,\}, \\ \ker A &= \{\, f \in \mathfrak{H} : \{f, 0\} \in A \,\}, \\ \operatorname{mul} A &= \{\, f' \in \mathfrak{H} : \{0, f'\} \in A \,\}. \end{aligned}$$

The closures of $\operatorname{dom} A$ and $\operatorname{ran} A$ are denoted by $\overline{\operatorname{dom}}\, A$ and $\overline{\operatorname{ran}}\, A$, respectively. When the relation A is closed, then $\ker A$ and $\operatorname{mul} A$ are automatically closed. A linear operator will be identified with its graph. A relation A has a formal inverse $A^{-1} = \{\, \{f', f\} : \{f, f'\} \in A \,\}$. Let A and B be linear relations in $\mathfrak{H}$. Then the product BA is the linear relation defined by

$$BA = \{\, \{f, g\} \in \mathfrak{H} \times \mathfrak{H} : \{f, \varphi\} \in A,\ \{\varphi, g\} \in B \text{ for some } \varphi \in \mathfrak{H} \,\}.$$

This definition agrees with the usual one for operators. For any $\lambda \in \mathbb{C}$ the relation $A - \lambda$ is defined by $A - \lambda = \{\, \{f, f' - \lambda f\} : \{f, f'\} \in A \,\}$. Let P be the orthogonal projection from $\mathfrak{H}$ onto $(\operatorname{mul} A)^{\perp}$. Then each $\{f, f'\} \in A$ can be uniquely decomposed as

$$\{f, f'\} = \{f, Pf'\} + \{0, (I - P)f'\}.$$

The linear relation

$$A_s = \{\, \{f, f'\} : \{f, f'\} \in A,\ f' = Pf' \,\} = \{\, \{f, Pf'\} : \{f, f'\} \in A \,\}$$

is called the (orthogonal) operator part of A: it is the graph of an operator from $\mathfrak{H}$ to $P\mathfrak{H} \subset \mathfrak{H}$. In the sense of multiplication of relations, A_s and A are related by $A_s = PA$. Define the linear relation A_∞ by

$$A_\infty = A \cap (\{0\} \times \mathfrak{H}).$$

Then the linear relation A admits the orthogonal decomposition

$$A = A_s \oplus A_\infty,$$

where the orthogonal sum is with respect to the inner product on $\mathfrak{H} \times \mathfrak{H}$.

The adjoint A^* of a linear relation A in $\mathfrak{H}$ is the linear relation in $\mathfrak{H}$, defined by

$$A^* = \{ \{f', f\} \in \mathfrak{H} \times \mathfrak{H} : \langle \{f', f\}, \{h, h'\} \rangle = 0, \, \{h, h'\} \in A \},$$

where

$$\langle \{f', f\}, \{h, h'\} \rangle = (f, h) - (f', h'), \quad \{f, f'\}, \{h, h'\} \in \mathfrak{H} \times \mathfrak{H}.$$

The adjoint A^* is automatically closed and linear.

The resolvent set $\rho(A)$ of a closed linear relation A in $\mathfrak{H}$ is defined by:

$$\rho(A) = \{ \lambda \in \mathbb{C} : (A - \lambda)^{-1} \in [\mathfrak{H}] \},$$

where $[\mathfrak{H}]$ denotes the set of all bounded linear operators on $\mathfrak{H}$ and $(A-\lambda)^{-1}$ is identified with its graph.

A linear relation A in $\mathfrak{H}$ is said to be symmetric if $(f', f) \in \mathbb{R}$ for all $\{f, f'\} \in A$, or, equivalently, if $A \subset A^*$. The relation A is said to be selfadjoint if $A = A^*$. If the relation A is selfadjoint, then $\overline{\operatorname{dom}} \, A = (\operatorname{mul} A)^{\perp}$ and A_s is a (densely defined) selfadjoint operator in $\overline{\operatorname{dom}} \, A$.

A linear relation A in a Hilbert space $\mathfrak{H}$ is said to be nonnegative, for short $A \geq 0$, if

$$(f', f) \geq 0, \quad \{f, f'\} \in A.$$

Clearly, every nonnegative relation is symmetric.

Let A be a nonnegative selfadjoint relation in a Hilbert space $\mathfrak{H}$. The square root $A^{\frac{1}{2}}$ of A is the unique nonnegative selfadjoint relation B in $\mathfrak{H}$ such that $B^2 = A$, as follows from the definition of the product.

3. Closed nonnegative forms

Let $\mathfrak{t} = \mathfrak{t}[\cdot, \cdot]$ be a nonnegative form in the Hilbert space $\mathfrak{H}$ with domain $\operatorname{dom} \mathfrak{t}$, cf. [7, Chapter VI]. The notation $\mathfrak{t}[h]$ will be used to denote $\mathfrak{t}[h, h]$, $h \in \operatorname{dom} \mathfrak{t}$. For the following definitions see [7, Chapter VI]. The inclusion $\mathfrak{t}_1 \subset \mathfrak{t}_2$ for nonnegative forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ is defined by

$$\operatorname{dom} \mathfrak{t}_1 \subset \operatorname{dom} \mathfrak{t}_2, \quad \mathfrak{t}_1[h] = \mathfrak{t}_2[h], \quad h \in \operatorname{dom} \mathfrak{t}_1.$$

The nonnegative form $\mathfrak{t}$ is closed if

$$h_n \to h, \quad \mathfrak{t}[h_n - h_m] \to 0, \quad h_n \in \operatorname{dom} \mathfrak{t}, \quad h \in \mathfrak{H}, \quad m, n \to \infty,$$

imply that $h \in \operatorname{dom} \mathfrak{t}$ and $\mathfrak{t}[h_n - h] \to 0$. The nonnegative form $\mathfrak{t}$ is closable if it has a closed extension; in this case the closure of $\mathfrak{t}$ is the smallest closed extension of $\mathfrak{t}$. The inequality $\mathfrak{t}_1 \geq \mathfrak{t}_2$ for nonnegative forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ is defined by

$$\operatorname{dom} \mathfrak{t}_1 \subset \operatorname{dom} \mathfrak{t}_2, \quad \mathfrak{t}_1[h] \geq \mathfrak{t}_2[h], \quad h \in \operatorname{dom} \mathfrak{t}_1.$$

In particular, $\mathfrak{t}_1 \subset \mathfrak{t}_2$ implies $\mathfrak{t}_1 \geq \mathfrak{t}_2$. If the forms $\mathfrak{t}_1$ and $\mathfrak{t}_2$ are closable, the inequality $\mathfrak{t}_1 \geq \mathfrak{t}_2$ is preserved by their closures. There is a one-to-one correspondence between all closed nonnegative forms $\mathfrak{t}$ in $\mathfrak{H}$ and all nonnegative selfadjoint relations A in $\mathfrak{H}$ via

$$\operatorname{dom} A \subset \operatorname{dom} \mathfrak{t},$$

and

$$\mathfrak{t}[f, g] = (A_s f, g), \quad f \in \operatorname{dom} A, \quad g \in \operatorname{dom} \mathfrak{t}.$$

Furthermore, let the nonnegative form $\mathfrak{t}$ and the nonnegative selfadjoint relation A be connected as above. If $\mathfrak{t} \geq 0$ or, equivalently, $A \geq 0$, then

$$\operatorname{dom} \mathfrak{t} = \operatorname{dom} A_s^{1/2},$$

and

$$\mathfrak{t}[f, g] = (A_s^{1/2} f, A_s^{1/2} g), \quad f, g \in \operatorname{dom} \mathfrak{t}.$$

4. Nonnegative selfadjoint extensions of nonnegative relations

Let S be a not necessarily closed symmetric relation in a Hilbert space $\mathfrak{H}$. Assume that S is nonnegative, so that also the closure $\operatorname{clos} S$ of S is nonnegative. Since the defect numbers of S and thus of $\operatorname{clos} S$, are equal, there exist selfadjoint extensions of S in $\mathfrak{H}$. In particular, one nonnegative selfadjoint extension can be constructed as follows. Let $\{f, f'\}, \{h, h'\} \in S$ and define $\mathfrak{s}[f, h] = (f', h)$, so that $\mathfrak{s}$ is a nonnegative form on $\operatorname{dom} \mathfrak{s} = \operatorname{dom} S$. The form $\mathfrak{s}$ is form-closable, cf. [7, VI Theorem 1.27]. The closure $\mathfrak{t}$ of the form $\mathfrak{s}$ is nonnegative (and is equal to the form obtained by starting with the closure of S) and gives rise to a nonnegative selfadjoint relation which is called the Friedrichs extension S_F of S. Let $\mathfrak{H}_a[S]$ be the completion of $\operatorname{dom} S$ with respect to the inner product $(f' + af, h)$, $\{f, f'\}, \{h, h'\} \in S$, where $a \in \mathbb{R}$ is chosen large enough to make the form positive definite.

Clearly, $\mathfrak{H}_a[S]$ and $\operatorname{dom} \mathfrak{t}$ coincide as sets. The Friedrichs extension is the only selfadjoint extension of S whose domain is contained in $\operatorname{dom} \mathfrak{t}$, cf. [4].

Now assume that S is nonnegative, so that its Friedrichs extension S_F is a nonnegative selfadjoint extension of S. The so-called Kreĭn-von Neumann extension S_N of S is defined by

$$S_N = ((S^{-1})_F)^{-1},$$

cf. [1], [5] for the case that S is not densely defined.

5. Inequalities for nonnegative linear relations

Assume that A_1 and A_2 are two nonnegative selfadjoint linear relations in the Hilbert space $\mathfrak{H}$. It is said that $A_1 \prec A_2$ if the following two conditions are satisfied:

$$\operatorname{dom} A_2^{\frac{1}{2}} \subset \operatorname{dom} A_1^{\frac{1}{2}},$$

and

$$\|A_{1s}^{\frac{1}{2}} h\| \leq \|A_{2s}^{\frac{1}{2}} h\|, \quad \text{for all} \quad h \in \operatorname{dom} A_2^{\frac{1}{2}}.$$

The relation $\prec$ is a partial order on the class of all nonnegative selfadjoint linear relations in $\mathfrak{H}$. In particular, $A_1 = A_2$ if and only if $A_1 \prec A_2$ and $A_2 \prec A_1$. It can be proved that $A_1 \prec A_2$ if and only if $\operatorname{dom} A_2 \subset \operatorname{dom} A_1^{\frac{1}{2}}$ and $\|A_{1s}^{\frac{1}{2}} h\| \leq (h', h)$ for all $\{h, h'\} \in A_2$.

6. Certain characterizations of nonnegative extensions

Given a nonnegative linear relation S in a Hilbert space $\mathfrak{H}$ define $EXT(S)$ as the class of all nonnegative selfadjoint linear relations A in $\mathfrak{H}$ such that $S \subset A$. Note that if S itself is a nonnegative selfadjoint linear relation in $\mathfrak{H}$ then $EXT(S) = \{S\}$.

Theorem 1 *Assume that S is a nonnegative linear relation in a Hilbert space $\mathfrak{H}$ and $A_1 \in EXT(S)$. If A_2 is a nonnegative selfadjoint linear relation in $\mathfrak{H}$ satisfying the following three conditions:*

(i) $\operatorname{dom} S \subset \operatorname{dom} A_2^{\frac{1}{2}}$*,*

(ii) $\|A_{2s}^{\frac{1}{2}} h\|^2 \leq (h', h)$ *for all* $\{h, h'\} \in S$*,*

(iii) $A_1 \prec A_2$,

then $A_2 \in EXT(S)$.

Corollary 2 *Assume that* S *is a nonnegative linear relation in a Hilbert space* $\mathfrak{H}$. *A nonnegative selfadjoint linear relation* A *extends* S *if and only if the following three conditions hold true:*

(i) $\operatorname{dom} S \subset \operatorname{dom} A^{\frac{1}{2}}$,

(ii) $\|A_s^{\frac{1}{2}} f\|^2 \leq (f', f)$ *for all* $\{f, f'\} \in S$,

(iii) $S_N \prec A$.

The result in Corollary 2 may be translated into the language of quadratic forms as follows.

Corollary 3 *Assume that* S *is a nonnegative linear relation in a Hilbert space* $\mathfrak{H}$, *and* $\mathfrak{t}$ *is a closed nonnegative quadratic form in* $\mathfrak{H}$. *The following two items are equivalent:*

(i) *there exists* $A \in EXT(S)$ *such that* $\mathfrak{t}_A = \mathfrak{t}$;

(ii) $\operatorname{dom} S \subset \operatorname{dom} \mathfrak{t}$, $\mathfrak{t}[f] \leq (f', f)$ *for all* $\{f, f'\} \in S$, *and* $\mathfrak{t}_N \leq \mathfrak{t}$.

7. Kreĭn's criteria

M.G. Kreĭn gave a complete description of the class $EXT(S)$ in the case S is a densely defined nonnegative operator (cf. [8, 9]). The next result extends this characterization to the case of nonnegative linear relations.

Theorem 4 *Assume that* S *is a nonnegative linear relation in a Hilbert space* $\mathfrak{H}$. *If* A_1, $A_2 \in EXT(S)$, *and* A *is a nonnegative selfadjoint linear relation in* $\mathfrak{H}$ *such that* $A_1 \prec A \prec A_2$ *then* $A \in EXT(S)$.

Corollary 5 *Let* S *be a nonnegative linear relation in a Hilbert space* $\mathfrak{H}$ *and* A *be a nonnegative selfadjoint linear relation in* $\mathfrak{H}$. *Then* A *belongs to* $EXT(S)$ *if and only if there exists* $B \in EXT(S)$ *such that* $S_N \prec A \prec B$.

References

[1] T. Ando and K. Nishio, *Positive selfadjoint extensions of positive symmetric operators*, Tôhoku Math. J., 22 (1970), 65–75.

[2] Yu.M. Arlinskiĭ, S. Hassi, Z. Sebestyén, and H.S.V. de Snoo, *On the class of extremal extensions of a nonnegative operator*, Oper. Theory: Adv. Appl. (B. Sz.-Nagy memorial volume), 127 (2001), 41–81.

[3] Yu.M. Arlinskiĭ and E.R. Tsekanovskiĭ, *On von Neumann's problem in extension theory of nonnegative operators*, Proc. Amer. Math. Soc., 131 (2003), 3143–3154.

[4] E.A. Coddington, *Extension theory of formally normal and symmetric subspaces*, Mem. Amer. Math. Soc., 134, 1973.

[5] E.A. Coddington and H.S.V. de Snoo, *Positive selfadjoint extensions of positive symmetric subspaces*, Math. Z., 159 (1978), 203–214.

[6] S. Hassi, A. Sandovici, H.S.V. de Snoo, and H. Winkler, *Form sums of nonnegative selfadjoint operators*, Acta Math. Hungar., 111 (2006), 81–105.

[7] T. Kato, Perturbation theory for linear operators, Springer-Verlag, Berlin, 1980.

[8] M.G. Kreĭn, *Theory of selfadjoint extensions of semibounded operators and its applications I*, Mat. Sb., 20 (1947), 431–495.

[9] M.G. Kreĭn, *Theory of selfadjoint extensions of semibounded operators and its applications II*, Mat. Sb., 21 (1947), 365–404.

[10] V. Prokaj and Z. Sebestyén, *On Friedrichs extensions of operators*, Acta Sci. Math. (Szeged), 62 (1996), 243–246.

[11] A. Sandovici, *On certain characterizations of nonnegative selfadjoint extensions*, in preparation.

[12] V. Prokaj and Z. Sebestyén, *On extremal positive operator extensions*, Acta Sci. Math. (Szeged), 62 (1996), 458–491.

[13] Z. Sebestyén and J. Stochel, *Restrictions of positive self-adjoint operators*, Acta Sci. Math. (Szeged), 55 (1991), 149–154.

[14] Z. Sebestyén and J. Stochel, *Characterizations of positive selfadjoint extensions*, Proc. Amer. Math. Soc., 135 (2007), 1389-1397.

[15] Z. Sebestyén, *Restrictions of positive operators*, Acta Sci. Math. (Szeged), 46 (1983), 299–301.

Author:

Adrian Sandovici
Department of Mathematics
University of Bacău
Str. Spiru Haret, nr. 8
600114 Bacău
Romania
email:*adrian.sandovici@gmail.com*