THE ROBSON CUBICS FOR MATRIX ALGEBRAS WITH INVOLUTION

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Abstract. J.C. Robson has investigated the ideal I_n of all polynomials in the free associative algebra $R\langle x\rangle$ over a noncommutative ring R generated by x and the n^2 entries of an $n\times n$ matrix $\alpha=(a_{ij})$, which are satisfied by α. He proved that I_n is finitely generated and found that four so called Robson cubics generate the ideal for $n=2$. The paper considers the ideal I_2 for matrix algebras with involution over a noncomutative ring and over a field of characteristic zero. The subspaces of the symmetric and skew-symmetric elements are studied separately and the explicit form of the Robson cubics is given in the considered cases. Some results are given for $n=3$ as well.

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1. Introduction

The Cayley-Hamilton theorem and the corresponding trace identity play a fundamental role in proving classical results about the polynomial and trace identities of the $n\times n$ matrix algebra $M_n(K)$ over a field K.

Accordingly, the importance of matrices over noncommutative rings is an evidence in the theory of PI-algebras and other branches of algebra as well (mainly structure theory of semisimple rings and quantum matrices).

The Cayley-Hamilton theorem has been extended by R. Paré and W. Schelter who have shown [2] that for any integer n an $n\times n$ matrix over a possibly noncommutative ring satisfies a monic polynomial with coefficients in that ring.

J.C. Robson has investigated in [6,7,8] the ideal I_n of all polynomials (including nonmonics) in the free associative algebra $R\langle x\rangle$ over a noncommutative ring R generated by x and the n^2 entries of an $n\times n$ matrix $\alpha=(a_{ij})$, which are satisfied by α.

Those polynomials we call the laws over R of a noncommutative $n \times n$ matrix α. These are not polynomial identities since the entries of α are allowed as coefficients in the laws and they vary with the choice of α. Of course, using the matrix units $\{e_{ij}\}$, one can eliminate these coefficients. Considering $n = 2$ we can replace a_{ij} by $(e_{1i}\alpha e_{j1} + e_{2i}\alpha e_{j2})$. This converts the law into a generalized polynomial identity in a single variable, with coefficients in $M_2(\mathbf{Z})$. It is a trivial generalized polynomial identity, since $M_2(\mathbf{Z})$ is itself a 2×2 matrix ring.

The general case considers the ideal of laws of a single $n\times n$ matrix extension over $\mathbf{Z}/(\mathrm{m})$ for some integer m, zero or not. The proof in [2] of the existence of a monic polynomial in I_n is inductive on n. In fact, given a monic polynomial of degree d in I_n, it gives a monic polynomial of degree $(d+1)^2$ in I_{n+1}. For 1×1 matrices there is no problem, $I_1 = (\alpha - a_{11})$. Therefore, there is a monic polynomial p of degree 4 in I_2. Writing

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for the generic 2×2 matrix, the polynomial is

$$p = ((\alpha)^2 - a\alpha - \alpha d - ad - cb)((\alpha)^2 - a\alpha - \alpha d + ad - bc). \qquad (**)$$

We recall that for

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

the ring over which we work is the free $\mathbf{Z}$-algebra $R = \mathbf{Z}\langle a, b, c, d\rangle$ and the polynomials are elements of $R\langle x\rangle$, which denotes the $\mathbf{Z}$-algebra freely generated by a, b, c, d and x.

A polynomial is in I_n if and only if all its homogeneous parts (i.e. parts each of whose terms has the same total degree as measured by the number of x's and a_{ij}'s in it) are in I_n, and so it is only necessary to look at homogeneous polynomials in I_n. Robson has shown that I_n is an insertive ideal (meaning that its homogeneous elements are closed under inner multiplication by a constant a_{ij} in some fixed position) and as such is finitely generated ([7] and [6, Theorem 2.3]).

Proposition 1 [6, Theorem 3.5] *Let $\alpha - x.1$ denote the $n \times n$ matrix*

$$\begin{pmatrix} a_{11}-x & a_{12} & ... & a_{1n} \\ a_{21} & a_{22}-x & ... & .... \\ ... & ... & ... & ... \\ a_{n1} & a_{2n} & ... & a_{nn}-x \end{pmatrix}.$$

The ideal I_n has a finite set of generators each of which is a polynomial in the entries of $\alpha - x.1$.

The minimal degree of polynomials in I_n remains unknown. However, for the case $n = 2$, Robson [6, Proposition 3.2] has found four polynomials of degree 3 (least possible) in I_2 and, knowing Proposition 3.5 of [8], has conjectured in [8] that these generate I_2 as an insertive ideal.

Proposition 2 [6, Proposition 3.2] *The set of homogeneous polynomials of degree 3 in I_2 is generated over $\mathbf{Z}/(m)$ for some integer m by the four polynomials*

$$\begin{aligned} r_a(x) &= x^3 - ax^2 - x^2a - xdx - xcb + cxb - cbx + adx \\ &+ xda + axa + acb - cab + cba - ada \\ &= (x-a)(x-d)(x-a) - cb(x-a) \\ &+ c(x-a)b - (x-a)cb, \\ r_b(x) &= bx^2 - xbx + x^2b - xab - dxb + dbx + xba - bxa \\ &- bdx + bda - dba + dab - bcb \\ &= b(x-d)(x-a) - (x-d)b(x-a) \\ &+ (x-d)(x-a)b - bcb, \\ r_c(x) &= cx^2 - xcx + x^2c - xdc - axc + acx + xcd - cxd \\ &- cax + cad - acd + adc - cbc \\ &= c(x-a)(x-d) - (x-a)c(x-d) \\ &+ (x-a)(x-d)c - cbc, \\ r_d(x) &= x^3 - dx^2 - x^2d - xax - xbc + bxc - bcx + dax \\ &+ xad + dxd + dbc - bdc + bcd - dad \\ &= (x-d)(x-a)(x-d) - bc(x-d) \\ &+ b(x-d)c - (x-d)bc. \end{aligned} \tag{1}$$

The polynomials $r_b(x)$ and $r_c(x)$ are not monic.

Pearson showed in [4, Corollary] that these four Robson cubics do indeed generate I_2 as an insertive ideal.

To illustrate the concept of insertive, we give some notation.

Let $w \in R\langle x\rangle$ have length m and $0 \leq t \leq m$. We can write $w = w_1 w_2$, where w_1 has length t. Let denote

$$\mu(m, t; g)w = w_1 g w_2.$$

Thus we see that if we inner multiply r_a by the constant b between the first and second letters of each word we get

$$\begin{aligned}\mu(3,1;b)r_a &= (x-a)b(x-d)(x-a) - cbb(x-a)\\ &+ cb(x-a)b - (x-a)bcb\end{aligned}$$

and it is easily checked that this polynomial is also in I_2.
Considering $(**)$ we could write

$$p = r_a(x)(x-d) - \mu(3,2;b)r_b.$$

Another example of a polynomial in I_2, given in [6], is

$$q = (bx - xb - bd + db)(bx - xb - ba + ab).$$

It could be written as $q = \mu(3,2;b)r_b - r_b b$.

2. Robson cubics for 2×2 matrices with involution

We specify the generators of I_2 in some special cases of an algebra with involution $*$ (an automorphism of order 2).

Proposition 3 [5, Proposition 1] *Considering the symmetric elements of $M_2(F,t)$ with noncommutative entries the Robson cubics for them are*

$$\begin{aligned}r_a(x) &= (x-a)(x-d)(x-a) - b^2(x-a)\\ &+ b(x-a)b - (x-a)b^2\\ &= [(x-a)(x-d) - b^2](x-a) + [b(x-a) - (x-a)b]b,\\ r_b(x) &= b(x-d)(x-a) - (x-d)b(x-a)\\ &+ (x-d)(x-a)b - b^3\end{aligned}$$

$$\begin{aligned}
&= [b(x-d)-(x-d)b](x-a)+[(x-d)(x-a)-b^2]b,\\
r_c(x) &= b(x-a)(x-d)-(x-a)b(x-d)\\
&+ (x-a)(x-d)b-b^3\\
&= [b(x-a)-(x-a)b](x-d)+[(x-a)(x-d)-b^2]b,\\
r_d(x) &= (x-d)(x-a)(x-d)-b^2(x-d)\\
&+ b(x-d)b-(x-d)b^2\\
&= [(x-d)(x-a)-b^2](x-d)+[b(x-d)-(x-d)b]b.
\end{aligned}$$

Theorem 1 *Considering the skew-symmetric elements of $M_2(F,t)$ with noncommutative entries the Robson cubics for them are*

$$\begin{aligned}
r_a(x) &= x^3+b^2x-bxb+xb^2=(x^2+b^2)x-(bx-xb)b,\\
r_b(x) &= bx^2-xbx+x^2b+b^3=b(x^2+b^2)-x(bx-xb).
\end{aligned}$$

Proof. For the skew-symmetric elements of $M_2(F,t)$ over a field F we have

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}^t=\begin{pmatrix} a & c\\ b & d\end{pmatrix}=\begin{pmatrix} -a & -b\\ -c & -d\end{pmatrix},$$

giving $b=-c, a=0=d$. Considering the entries to be noncommutative and substituting in (1) we get

$$\begin{aligned}
r_a(x) &= x^3+b^2x-bxb+xb^2,\\
r_b(x) &= bx^2-xbx+x^2b+b^3,\\
r_c(x) &= -bx^2+xbx-x^2b-b^3=-r_b(x),\\
r_d(x) &= x^3+b^2x-bxb+xb^2=r_a(x).
\end{aligned}$$

Theorem 2 *Let's consider $M_2(F,*)$, where $*$ is the symplectic involution and its symmetric elements but with noncommutative entries. The Robson cubics for them turn into one first degree polynomial, namely*

$$r(x)=x-a.$$

Proof. We consider the symmetric elements of $M_2(F,*)$ for $*$ being the symplectic involution. The equality

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix}^*=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$$

gives $b = c = 0, a = d$. Then (1) shows

$$\begin{aligned} r_a(x) &= (x-a)^3, \\ r_b(x) &= r_c(x) = 0, \\ r_d(x) &= (x-a)^3. \end{aligned}$$

The degree of the polynomial $r_a(x)$ is really 1, i.e. $r(x) = x - a$.

Proposition 4 [5, Proposition 6] *Let's consider $M_2(F, *)$, where $*$ is the symplectic involution and its skew-symmetric elements but with noncommutative entries. The Robson cubics for them are*

$$\begin{aligned} r_a(x) &= (x-a)(x+a)(x-a) - cb(x-a) \\ &+ c(x-a)b - (x-a)cb \\ &= [(x-a)(x+a) - cb](x-a) + [c(x-a) - (x-a)c]b, \\ r_b(x) &= b(x+a)(x-a) - (x+a)b(x-a) \\ &+ (x+a)(x-a)b - bcb \\ &= [b(x+a) - (x+a)b](x-a) + [(x+a)(x-a) - bc]b, \\ r_c(x) &= c(x-a)(x+a) - (x-a)c(x+a) \\ &+ (x-a)(x+a)c - cbc \\ &= [c(x-a) - (x-a)c](x+a) + [(x-a)(x+a) - cb]c, \\ r_d(x) &= (x+a)(x-a)(x+a) - bc(x+a) \\ &+ b(x+a)c - (x+a)bc \\ &= [(x+a)(x-a) - bc](x+a) + [b(x+a) - (x+a)b]c. \end{aligned}$$

In [1] D. La Mattina and P. Misso study some associative algebras with involution investigating their polynomial growth. We will consider here some of these algebras.

The first case will be a noncommutative one.

Let R be a noncommutative ring and

$$M1(R) = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}; a, b \in R\}.$$

Theorem 3 *A law for a matrix of $M1(R)$ is $r = (x-a)^2$. For a scalar matrix $r(x) = x - a$.*

Proof. Direct computations.

Remark 1 *We point that over a field F of characteristic zero the Robson cubics* (1) *for $M_2(F,*)$ turn into one second degree polynomial given by the Cayley-Hamilton theorem, namely*

$$r(x) = (x-a)(x-d) - bc.$$

Really in this case (1) gives

$$\begin{aligned} r_a(x) &= (x-a)(x-d)(x-a) - bc(x-a), \\ r_b(x) &= b(x-d)(x-a) - b^2c, \\ r_c(x) &= c(x-a)(x-d) - bc^2, \\ r_d(x) &= (x-d)(x-a)(x-d) - bc(x-d). \end{aligned}$$

The Cayley-Hamilton theorem for a 2×2 matrix A means that $A^2 - \mathrm{trA}.A + \det A.E = 0$. If $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have $\alpha^2 - (a+d)\alpha + ad - bc = 0$. This equation could be written as $r(x) = (x-a)(x-d)-bc = (x-d)(x-a)-bc = 0$. Thus $r_a(x) = r(x)(x-a)$, $r_d(x) = r(x)(x-d)$, $r_b(x) = br(x)$ and $r_c(x) = cr(x)$.

3. Laws for 3×3 matrices

Now we give some evidence in the case $n = 3$ for a field F. The first study of the noncommutative case was done in [3].The results there provide further evidence of the tantalizing complexity of even these small matrices.

We start with the matrix algebra $M_3(F)$ over a field F of characteristic zero. We try to find some of the polynomials of the algebra $F\langle x\rangle$ generated by x and the 9 entries of a 3×3 matrix α, which are satisfied by α.

Using the Newton's formulas [9, p.18] the Cayley-Hamilton theorem gives that for a matrix $A \in M_3(F)$

$$A^3 - \alpha_1 A^2 + \alpha_2 A - \alpha_3 E = 0. \tag{2}$$

We have

$$\begin{aligned} \alpha_1 &= \mathrm{tr}A \\ 2\alpha_2 &= \alpha_1 \mathrm{tr}A - \mathrm{tr}A^2 = \mathrm{tr}^2 A - \mathrm{tr}A^2 \\ 3\alpha_3 &= \alpha_2 \mathrm{tr}A - \alpha_1 \mathrm{tr}A^2 + \mathrm{tr}A^3. \end{aligned} \tag{3}$$

Theorem 4 *For a skew-symmetric matrix*

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \in M_3(F,t)$$

we have

$$r_{ss}(x) = x^3 + (a^2 + b^2 + c^2)x \in F\langle x \rangle.$$

Proof: We see that $\mathrm{tr}A = 0$, $\mathrm{tr}A^2 = -2(a^2 + b^2 + c^2)$ and $\mathrm{tr}A^3 = 0$. Thus we get

$$A^3 - \frac{1}{2}(\mathrm{tr}A^2)A - \frac{1}{3}(\mathrm{tr}A^3)E = 0, \text{i.e.}$$
$$r_{ss}(x) = x^3 + (a^2 + b^2 + c^2)x.$$

Remark 2 *For a symmetric matrix $A = (a_{ij}) \in M_3(F,t)$ the corresponding polynomial is too long to be written. We give only parts of its coefficients, i.e.*

$$\begin{aligned} \mathrm{tr}A &= a_{11} + a_{22} + a_{33}, \\ \mathrm{tr}A^2 &= 2(a_{11}^2 + a_{22}^2 + a_{33}^2 + a_{12}^2 + a_{13}^2 + a_{23}^2), \\ \mathrm{tr}A^3 &= a_{11}^3 + a_{22}^3 + a_{33}^3 + 6a_{12}a_{13}a_{23} \\ &+ 3a_{12}^2(a_{11} + a_{22}) + 3a_{13}^2(a_{11} + a_{33}) + 3a_{23}^2(a_{22} + a_{33}). \end{aligned}$$

In [1] three 3×3 matrix algebras are considered.

Let

$$M2(F) = \{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} ; a, b, c \in F \}.$$

The algebra $M2(F)$ is endowed with the involution

$$\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & -b & c \\ 0 & a & -b \\ 0 & 0 & a \end{pmatrix}.$$

Theorem 5 *A law for $M2(F)$ is $r(x) = (x - a)^3$. For the symmetric elements of $(M2(F), *)$ we have $r_s(x) = (x - a)^2$ and for the skew-symmetric elements $r_{ss}(x) = x^3$.*

Proof. For a matrix $A \in M2(F)$ we get $\mathrm{tr}A = 3a$, $\mathrm{tr}A^2 = 3a^2$ and $\mathrm{tr}A^3 = 3a^3$. Thus (2) and (3) give $\alpha_1 = 3a$, $\alpha_2 = 3a^2$, $\alpha_3 = a^3$ and

$$r(x) = x^3 - 3ax^2 + 3a^2x - a^3 = (x-a)^3.$$

In the symmetric case $(b = 0)$ we get really $r_s(x) = (x-a)^2$ while in the skew-symmetric one $(a = c = 0)$ we have $r_{ss}(x) = x^3$.

Let

$$M3(F) = \{\begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & a \end{pmatrix}; a, b, c, d \in F\}.$$

The algebra $M3(F)$ is endowed with the involution

$$\begin{pmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & a \end{pmatrix}^* = \begin{pmatrix} a & -b & c \\ 0 & 0 & -d \\ 0 & 0 & a \end{pmatrix}.$$

Theorem 6 *A law for $M3(F)$ is $r(x) = (x-a)^2x$. For the symmetric elements of $(M3(F), *)$ we have $r_s(x) = (x-a)^2$ and for the skew-symmetric elements $r_{ss}(x) = x^3$.*

Proof. For a matrix $A \in M3(F)$ we get $\mathrm{tr}A = 2a$, $\mathrm{tr}A^2 = 2a^2$ and $\mathrm{tr}A^3 = 2a^3$. Thus (2) and (3) give $\alpha_1 = 2a$, $\alpha_2 = a^2$, $\alpha_3 = 0$ and

$$r(x) = x^3 - 2ax^2 + a^2x = (x-a)^2x.$$

In the symmetric case $(b = c = d = 0)$ we get really $r_s(x) = (x-a)^2$ while in the skew-symmetric one $(a = c = 0)$ we have $r_{ss}(x) = x^3$.

At the end we consider

$$M4(F) = \{\begin{pmatrix} 0 & b & c \\ 0 & a & d \\ 0 & 0 & 0 \end{pmatrix}; a, b, c, d \in F\}.$$

The algebra $M4(F)$ is endowed with the involution

$$\begin{pmatrix} 0 & b & c \\ 0 & a & d \\ 0 & 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & -b & c \\ 0 & a & -d \\ 0 & 0 & 0 \end{pmatrix}.$$

Theorem 7 *A law for $M4(F)$ is $r(x) = (x-a)x^2$. For the symmetric elements of $(M4(F), *)$ we have $r_s(x) = (x-a)x$ and for the skew-symmetric elements $r_{ss}(x) = x^3$.*

Proof. For a matrix $A \in M4(F)$ we get $\mathrm{tr}A = a$, $\mathrm{tr}A^2 = a^2$ and $\mathrm{tr}A^3 = a^3$. Thus (2) and (3) give $\alpha_1 = a$, $\alpha_2 = 0$, $\alpha_3 = 0$ and

$$r(x) = x^3 - ax^2 = (x-a)x^2.$$

In the symmetric case ($b = d = 0$) we get $r_s(x) = (x-a)x$. In the skew-symmetric case ($a = c = 0$) we have $r_{ss}(x) = x^3$.

Remark 3 *If we consider $M3(F)$ and $M4(F)$ with involution defined by reflecting a matrix along its secondary diagonal the corresponding laws are really cubics, i.e.*

$$\deg r(x) = \deg r_s(x) = \deg r_{ss}(x) = 3.$$

In [8] *J.C. Robson found laws of degree* 7 *for* $M_3(R)$ *over a noncommutative ring R. They are four and each of them has 1156 terms.*

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