$G-N$-QUASIGROUPS AND FUNCTIONAL EQUATIONS ON QUASIGROUPS

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Abstract. Based on $G-n$-quasigroups we give straightforward methods to solve the functional equations of generalized associativity, cyclic associativity and bisymmetry on quasigroups.

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In this paper we solve functional equations of generalized associativity (1), cyclic associativity (2) and bisymmetry (or mediality) (3).

$$\alpha_1(\alpha_2(x,y),z) = \alpha_3(x,\alpha_4(y,z)) \quad (1)$$

$$\alpha_1(x,\alpha_2(y,z)) = \alpha_3(y,\alpha_4(z,x)) = \alpha_5(z,\alpha_6(x,y)) \quad (2)$$

$$\alpha_1(\alpha_2(x,y),\alpha_3(z,u)) = \alpha_4(\alpha_5(x,z),\alpha_6(y,u)) \quad (3)$$

where x, y, z, u are taken from an arbitrary set A and α_i are quasigroup operations on A.

The method used is an example of the application of our results developed in [3].

All three equations have been investigated by Aczél, Belousov and Hosszú [1] and Belousov [2]. The used methods to solve these equations have an involved part: the proof that all quasigroups (A, α_i) are isotopic to the same group $(A, +)$. Our method reduces the proofs to a routine calculation.

For notions and notations see [3].

1. The Equation of Generalized Associativity

Theorem 1. *The set of all solutions of the functional equation of generalized associativity over the set of quasigroup operations on an arbitrary set A is described by the relations*

$$\begin{aligned}&\alpha_1(x,y) = F_1(x)+F_2(y), \alpha_2(x,y) = F_1^{-1}(F_3(x)+F_4(y)),\\ &\alpha_3(x,y) = F_3(x)+F_5(y), \alpha_4(x,y) = F_5^{-1}(F_4(x)+F_2(y)),\end{aligned} \quad (4)$$

where $(A,+)$ is an arbitrary group and $F_1,\dots,F_5$ are arbitrary substitutions of the set A.

Proof. It is obvious that all (A,α_i) defined by relations (4) are quasigroups isotopic to group $(A,+)$. Substituting these values of the functional variables in (1) we obtain for the both sides the same expression.

Conversely, let $\alpha_1,\dots,\alpha_4$ be four quasigroup operations on A and forming a solution of the functional equation (1). We define $\alpha : A^3 \to A$ by

$$\alpha(x,y,z) = \alpha_1(\alpha_2(x,y),z) = \alpha_3(x,\alpha_4(y,z)) \tag{5}$$

It is easy to prove that (A,α) is a 3-quasigroup. Moreover, cf. Theorem 6. [3] (A,α) is a G-quasigroup, i.e. $\alpha(x,y,z) = T_1(x)+T_2(y)+T_3(z)$, where $(A,+)$ is a group and T_i are translations by $a=(a_1^3)\in A^3$ in (A,α). The zero element of $(A,+)$ is $0=T_1(a_1)=T_2(a_2)=T_3(a_3)$.

Putting $z=a_3$ in (5) we obtain

$$\alpha_1(\alpha_2(x,y),a_3) = \alpha_3(x,\alpha_4(y,a_3)) = T_1(x)+T_2(y) \tag{6}$$

Since (A,α_1) and (A,α_4) are quasigroups, the mappings $f(x)=\alpha_1(x,a_3)$ and $g(x)=\alpha_4(x,a_3)$ are substitutions of A.

From (6) we get

$$\alpha_2(x,y) = f^{-1}(T_1(x)+T_2(y)$$

and

$$\alpha_3(x,y) = T_1(x)+T_2(g^{-1}(y)).$$

If in equality (5) we put $x=a_1$ then we get

$$\alpha_1(\alpha_2(a_1,y),z) = \alpha_3(a_1,\alpha_4(y,z)) = T_2(y)+T_3(z) \tag{7}$$

The mappings $h(x)=\alpha_2(a_1,x)$ and $u(x)=\alpha_3(a_1,x)$ are substitutions of A ((A,α_2) and (A,α_3) are quasigroups). Therefore, from (7) we have

$$\alpha_1(y,z) = T_2(h^{-1}(y))+T_3(z)$$

and

$$\alpha_4(y,z) = u^{-1}(T_2(y)+T_3(z).$$

Now, from

$$f(h(x)) = \alpha_1(h(x),a_3) = \alpha_1(\alpha_2(a_1,x),a_3) = \alpha(a_1,x,a_3) = T_2(x)$$

we obtain $T_2 \circ h^{-1} = f$ and from

$$u(g(x)) = \alpha_3(a_1, g(x)) = \alpha_3(a_1, \alpha_4(x, a_3)) = \alpha(a_1, x, a_3) = T_2(x)$$

we get $T_2 \circ g^{-1} = u$.

Taking into account the above results we have

$$\begin{array}{ll} \alpha_1(x,y) = f(x) + T_3(y), & \alpha_2(x,y) = f^{-1}(T_1(x) + T_2(y)), \\ \alpha_3(x,y) = T_1(x) + u(y), & \alpha_4(x,y) = u^{-1}(T_2(x) + T_3(y)). \end{array} \tag{8}$$

Thus, it follows from (8) that any solution of the equation (1) has the form (4).

2. The Equation of Generalized Cyclic Associativity

Theorem 2. *The set of all solutions of the functional equation of generalized cyclic associativity over the set of quasigroup operations on an arbitrary set A is described by the relations*

$$\begin{array}{ll} \alpha_1(x,y) = F_1(x) + F_2(y), & \alpha_2(x,y) = F_2^{-1}(F_3(x) + F_4(y)), \\ \alpha_3(x,y) = F_3(x) + F_5(y), & \alpha_4(x,y) = F_5^{-1}(F_4(x) + F_1(y)), \\ \alpha_5(x,y) = F_4(x) + F_6(y), & \alpha_6(x,y) = F_6^{-1}(F_1(x) + F_3(y)) \end{array} \tag{9}$$

where $(A,+)$ is an arbitrary commutative group and $F_1, \ldots, F_6$ are arbitrary substitutions of the set A.

Proof. Let $\alpha_1, \ldots, \alpha_6$ be six quasigroup operations on A and forming a solution of equation (2). We define $\alpha : A^3 \to A$ by

$$\alpha(x,y,z) = \alpha_1(x, \alpha_2(y,z)) = \alpha_3(y, \alpha_4(z,x)) = \alpha_5(z, \alpha_6(x,y)) \tag{10}$$

It is obvious that (A, α) is a 3-quasigroup.

From $\alpha(x,y,z) = \alpha_1(x, \alpha_2(y,z))$ it follows immediately that condition $D_{2,3}$ holds in (A,α), $\alpha(x,y,z) = \alpha_3(y, \alpha_4(z,x))$ implies that condition D_{1-3} holds in (A,α) and from $\alpha(x,y,z) = \alpha_5(z, \alpha_6(x,y))$ we obtain that condition $D_{1,2}$ holds in (A,α). Therefore, cf. Theorem 9 [3] (A,α) is a G_a-quasigroup, i.e. $\alpha(x,y,z) = T_1(x) + T_2(y) + T_3(z)$, where $(A,+)$ is a commutative group, T_i are translations by an arbitrary element $a = (a_1^3) \in A^3$ in (A,α) and $T_1(a_1) = T_2(a_2) = T_3(a_3) = 0$ - zero element of $(A,+)$.

From (10), for $x = a_1$ we obtain

$$\alpha_1(a_1, \alpha_2(y,z)) = \alpha_3(y, \alpha_4(z,a_1)) = \alpha_5(z, \alpha_6(a_1,y)) = T_2(y) + T_3(z).$$

Therefore

$$\begin{array}{ll}\alpha_2(y,z)=f^{-1}(T_2(y)+T_3(z)), & \text{where} \quad f(x)=\alpha_1(a_1,x),\\ \alpha_3(y,z)=T_2(y)+T_3(g^{-1}(z)), & \text{where} \quad g(x)=\alpha_4(x,a_1),\\ \alpha_5(z,y)=T_3(z)+T_2(h^{-1}(y)), & \text{where} \quad h(x)=\alpha_6(a_1,x).\end{array}$$

Putting $z=a_3$ in (10) we get

$$\alpha_1(x,\alpha_2(y,a_3))=\alpha_5(a_3,\alpha_6(x,y))=T_1(x)+T_2(y).$$

In consequence

$$\begin{array}{ll}\alpha_1(x,z)=T_1(x)+T_2(u^{-1}(y)), & \text{where} \quad u(x)=\alpha_2(x,a_3),\\ \alpha_6(x,y)=v^{-1}(T_1(x)+T_2(y)), & \text{where} \quad v(x)=\alpha_5(a_3,x).\end{array}$$

If in (10) we put $y=a_2$ then we obtain $\alpha_3(a_2,\alpha_4(z,x))=T_1(x)+T_3(z)$ hence $\alpha_4(z,x)=w^{-1}(T_1(x)+T_3(z))$ where $w(x)=\alpha_3(a_2,x)$.

Now,

$$\begin{array}{ll}f(u(x))=\alpha_1(a_1,\alpha_2(x,a_3))=T_2(x), & \text{i.e.} \quad f=T_2\circ u^{-1},\\ w(g(x))=\alpha_3(a_2,\alpha_4(x,a_1))=T_3(x), & \text{i.e.} \quad w=T_3\circ g^{-1} \text{ and}\\ v(h(x))=\alpha_5(a_3,\alpha_6(a_1,x))=T_2(x), & \text{i.e.} \quad v=T_2\circ h^{-1}.\end{array}$$

In conclusion,

$$\begin{array}{ll}\alpha_1(x,y)=T_1(x)+f(y), & \alpha_2(x,y)=f^{-1}(T_2(x)+T_3(y))\\ \alpha_3(x,y)=T_2(x)+w(y), & \alpha_4(x,y)=w^{-1}(T_1(x)+T_3(y)),\\ \alpha_5(x,y)=v(x)+T_3(y), & \alpha_6(x,y)=v^{-1}(T_1(x)+T_2(y))\end{array} \tag{11}$$

Thus, it follows from (11) that any solution of the equation (2) has the form (9).

The converse is obvious.

3. The Equation of Generalized Bisymmetry

Theorem 3. *The set of all solutions of the functional equation of generalized bisymmetry over the set of quasigroup operations on an arbitrary set A is described by the relations*

$$\begin{array}{ll}\alpha_1(x,y)=F_1(x)+F_2(y), & \alpha_2(x,y)=F_1^{-1}(F_3(x)+F_4(y)),\\ \alpha_3(x,y)=F_2^{-1}(F_5(x)+F_6(y)), & \alpha_4(x,y)=F_7(x)+F_8(y),\\ \alpha_5(x,y)=F_7^{-1}(F_3(x)+F_5(y)), & \alpha_6(x,y)=F_8^{-1}(F_4(x)+F_6(y)),\end{array} \tag{12}$$

where $(A, +)$ *is an arbitrary commutative group and* $F_1, \dots, F_8$ *are arbitrary substitutions of the set* A.

Proof. Let $\alpha_1, \dots, \alpha_6$ be six quasigroup operations on A and forming a solution of equation (3). We define $a : A^4 \to A$ by

$$\alpha(x, y, z, u) = \alpha_1(\alpha_2(x, y), \alpha_3(z, u)) = \alpha_4(\alpha_5(x, z), a_6(y, u)) \tag{13}$$

It is obvious that (A, α) is 4-quasigroup. According to Theorem 11 [3] (A, α) is a G_a-quasigroup, i.e.

$\alpha(x, y, z, u) = T_1(x) + T_2(y) + T_3(z) + T_4(u)$, where $(A, +)$ is a commutative group with zero element $0 = T_1(a_1) = T_2(a_2) = T_3(a_3) = T_4(a_4)$, T_i being translations by $a = (a_1^4) \in A^4$ in (A, α).

Putting $z = a_3$ and $u = a_4$ in (13) we get

$$\alpha_1(\alpha_2(x, y), \alpha_3(a_3, a_4)) = \alpha_4(\alpha_5(x, a_3), \alpha_6(y, a_4)) = T_1(x) + T_2(y) \tag{14}$$

Since (A, α_1), (A, α_5) and (A, α_6) are quasigroups the mappings $f(x) = \alpha_1(x, \alpha_3(a_3, a_4))$, $f_1(x) = \alpha_5(x, a_3)$ and $f_2(x) = \alpha_6(x, a_4)$ are substitutions of the set A. From (14) we obtain $\alpha_2(x, y) = f^{-1}(T_1(x) + T_2(y))$ and $\alpha_4(x, y) = T_1(f_1^{-1}(x)) + T_2(f_2^{-1}(y))$. For $x = a_1$ and $y = a_2$ in (13) we have

$$\alpha_1(\alpha_2(a_1, a_2), \alpha_3(z, u)) = T_3(z) + T_4(u)$$

and thus

$$\alpha_3(z, u) = g^{-1}(T_3(z) + T_4(u)), \quad \text{where} \quad g(x) = \alpha_1(\alpha_2(a_1, a_2), x).$$

If we put $y = a_2$ and $z = a_3$ in (13) then we get

$$\alpha_1(\alpha_2(x, a_2), \alpha_3(a_3, u)) = T_1(x) + T_4(u)$$

Hence $\alpha_1(x, u) = T_1(g_1^{-1}(x)) + T_4(g_2^{-1}(u))$. Putting $y = a_2$ and $u = a_4$ in (13) we obtain $\alpha_4(\alpha_5(x, z), \alpha_6(a_2, a_4)) = T_1(x) + T_3(z)$ and then $\alpha_5(x, z) = h^{-1}(T_1(x) + T_3(z))$ where $h(x) = \alpha_4(x, \alpha_6(a_2, a_4))$.

Finally, if we put $x = a_1$ and $z = a_3$ in (13) then we have

$$\alpha_4(\alpha_5(a_1, a_3),\ \alpha_6(y, u)) = T_2(y) + T_4(u)$$

and thus $\alpha_6(y, u) = h_1^{-1}(T_2(y) + T_4(u))$ for $h_1(x) = \alpha_4(\alpha_5(a_1, a_3), x)$.

Now, $f(g_1(x)) = \alpha_1(g_1(x), \alpha_3(a_3, a_4)) = \alpha_1(\alpha_2(x, a_2)), \alpha_3(a_3, a_4)) = T_1(x)$ implies that $T_1 \circ g_1^{-1} = f$,

$$g(g_2(x)) = \alpha_1(\alpha_2(a_1, a_2), g_2(x)) = \alpha_1(\alpha_2(a_1, a_2), \alpha_3(a_3, x)) = T_4(x)$$

implies that $T_4 \circ g_2^{-1} = g$,

$$h(f_1(x)) = \alpha_4(f_1(x), \alpha_6(a_2, a_4)) = \alpha_4(\alpha_5(x, a_3), \alpha_6(a_2, a_4)) = T_1(x)$$

implies that $T_1 \circ f_1^{-1} = h$ and

$$h_1(f_2(x)) = \alpha_4(\alpha_5(a_1, a_3), f_2(x)) = \alpha_4(\alpha_5(a_1, a_3), \alpha_6(x, a_4)) = T_2(x)$$

implies that $T_2 \circ f_2^{-1} = h_1$.

Taking into account the above results we have

$$\begin{array}{ll} \alpha_1(x,y) = f(x) + g(y), & \alpha_2(x,y) = f_1^{-1}(T_1(x) + T_2(y)), \\ \alpha_3(x,y) = g^{-1}(T_3(x) + T_4(y)), & \alpha_4(x,y) = h(x) + h_1(y), \\ \alpha_5(x,y) = h^{-1}(T_1(x) + T_3(y))), & \alpha_6(x,y) = h_1^{-1}(T_2(x) + T_4(y)). \end{array} \tag{15}$$

Thus, it follows from (15) that any solution of the equation (3) has the form (12).

The converse is clear.

References

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