ON ISOMORPHIC COMMUTATIVE GROUP ALGEBRAS OF ABELIAN GROUPS WITH $p^{\omega+n}$-PROJECTIVE QUOTIENTS

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Abstract. Let R be a commutative unitary ring of prime characteristic p and G an Abelian p-group such that $G^{p^{\omega+n}}$ is either torsion-complete or totally projective and $G/G^{p^{\omega+n}}$ is $p^{\omega+n}$-projective. We prove that the group algebra RG over R determines up to isomorphism G, that is, if RH and RG are isomorphic as R-algebras for another group H then H is isomorphic to G.

This extends classical results due to May Proc. Amer. Math. Soc., 1979 and Proc. Amer. Math. Soc., 1988 as well as results of ours in Proc. Amer. Math. Soc., 2002, Acta Math. Vietnam., 2004 and Acta Sci. Math. (Szeged), 2007.

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1. Introduction

Throughout the present paper, unless specifically stated otherwise, by the term "group" we will mean "*an Abelian p-group*", where p is a prime fixed for the duration, written multiplicatively as is customary when discussing group rings. Moreover, suppose everywhere in the text that R is a commutative ring with identity element (such a ring is often called *unitary*) and of prime characteristic p. As usual, RG will always denote the group algebra of G over R with a group of units $U(RG)$ and its subgroup consisting only of normalized units designated by $V(RG)$. If $U(R)$ is the unit group of R, the decomposition formula $U(RG) = V(RG) \times U(R)$ holds.

For any subgroup A of G, the symbol $I(RG; A)$ shall denote the relative augmentation ideal of RG with respect to A. All other notations and notions from Abelian group theory are standard and follow essentially those from [8]. For instance, $G^{p^\omega} = \cap_{i<\omega} G^{p^i}$ denotes the first Ulm subgroup of G consisting of all elements with infinite height in G, where $G^{p^i} = \{g^{p^i} \mid g \in G\}$ is the p^i-th power subgroup of G consisting of all elements with heights in G no less than i. Moreover, if α is an arbitrary ordinal, $G^{p^\alpha} = (G^{p^{\alpha-1}})^p$ if α is isolated and $G^{p^\alpha} = \cap_{\beta<\alpha} G^{p^\beta}$ when α is limit. By analogy, $R^{p^\omega} = \cap_{i<\omega} R^{p^i}$ denotes the first Ulm subring of R consisting of all elements with infinite height in R, where $R^{p^i} = \{r^{p^i} \mid r \in R\}$ is the p^i-th power subring of R consisting of all elements with heights in R at least i. Moreover, if α is an arbitrary ordinal, $R^{p^\alpha} = (R^{p^{\alpha-1}})^p$ if α is isolated and $R^{p^\alpha} = \cap_{\beta<\alpha} R^{p^\beta}$ when α is limit. Traditionally, $G[p^n] = \{g : g^{p^n} = 1\}$ denotes the p^n-socle of G whenever n is a natural number.

One of the main questions in the theory of commutative group algebras is the following one:

Problem. Does it follow that for any groups G and H the R-isomorphism $RG \cong RH$ will imply $G \cong H$?

This query seems to be practically insurmountable in the general case, so even its resolution for a concrete class of groups is of some interest and importance. Several major classes of groups are determined by their group algebra. Some more attractive of them are these (see [11], [12] and [4], [7] respectively):

Theorem A (May, 1979-1988). *Let G be a simply presented group and $RH \cong RG$ as R-algebras for another group H. Then $H \cong G$.*

Theorem B (Danchev, 2002). *Let G be a torsion-complete group and $RH \cong RG$ as R-algebras for some other group H. Then $H \cong G$.*

For the next two statements, as usual, $\mathbf{F}_p$ denotes the simple field of p-elements; notice that it is finite of prime characteristic p and thus it is perfect.

Theorem C (Danchev, 2004). *Let G be a $p^{\omega+1}$-projective group and $\mathbf{F}_p H \cong \mathbf{F}_p G$ as $\mathbf{F}_p$-algebras for another group H. Then $H \cong G$.*

The last assertion was generalized to

Theorem D (Danchev, 2007). *Let G be a separate $p^{\omega+1}$-totally projective group. If $\mathbf{F}_p H \cong \mathbf{F}_p G$ as $\mathbf{F}_p$-algebras for some other group H, then $H \cong G$.*

For completeness of the exposition, we recall that G is said to be *a separate $p^{\omega+n}$-totally projective group* for some positive integer n if G^{p^ω} is totally projective and there exists $P \leq G[p^n]$ with $P \cap G^{p^\omega} = 1$ and $G/(P \times G^{p^\omega})$ is a direct sum of cyclic groups. Notice that the groups G can be equivalently redefined as follows: There is a nice p-bounded subgroup P of G such that $P \cap G^{p^\omega} = 1$ and G/P is totally projective.

The purpose of this work is to relate and extend most of these investigations. In order to do that we shall utilize a significant criterion for the isomorphism of groups belonging to a special sort, established by Keef in [10] (see [9] as well).

2. Chief Results

Before stating and proving our central theorem, we need a few technicalities, starting with

Lemma 1. *For every ordinal α the following formula holds*

(i) $(RG)^{p^\alpha} = R^{p^\alpha} G^{p^\alpha}$;

(ii) $V^{p^\alpha}(RG) = V(R^{p^\alpha} G^{p^\alpha})$.

Proof. We shall use a transfinite induction on α.

(i) For $\alpha = 1$ the result is obvious. If now α is isolated, we have $(RG)^{p^\alpha} = ((RG)^{p^{\alpha-1}})^p = (R^{p^{\alpha-1}} G^{p^{\alpha-1}})^p = (R^{p^{\alpha-1}})^p (G^{p^{\alpha-1}})^p = R^{p^\alpha} G^{p^\alpha}$. Otherwise, when α is a limit ordinal, we easily obtain that $(RG)^{p^\alpha} = \cap_{\beta<\alpha}(RG)^{p^\beta} = \cap_{\beta<\alpha}(R^{p^\beta} G^{p^\beta}) = (\cap_{\beta<\alpha} R^{p^\beta})(\cap_{\beta<\alpha} G^{p^\beta}) = R^{p^\alpha} G^{p^\alpha}$, as stated.

(ii) If P is any unitary commutative ring of prime characteristic p, then it is not hard to verify that $U^{p^\alpha}(P) = U(P^{p^\alpha})$. Consequently, by what we have shown in the previous point, $U^{p^\alpha}(RG) = U((RG)^{p^\alpha}) = U(R^{p^\alpha} G^{p^\alpha})$ since RG is a commutative unitary ring with characteristic p. But $U^{p^\alpha}(RG) = (V(RG) \times U(R))^{p^\alpha} = V^{p^\alpha}(RG) \times U^{p^\alpha}(R)$ and $U(R^{p^\alpha} G^{p^\alpha}) = V(R^{p^\alpha} G^{p^\alpha}) \times U(R^{p^\alpha})$. Thus $V^{p^\alpha}(RG) = V(R^{p^\alpha} G^{p^\alpha})$, as expected. □

Proposition 2. *Suppose that RG and RH are R-isomorphic. Then, for each ordinal α,*

(j) RG^{p^α} and RH^{p^α} are R-isomorphic;

(jj) $R(G/G^{p^\alpha})$ and $R(H/H^{p^\alpha})$ are R-isomorphic.

Proof. It is obviously true that $RG \cong RH$ secures that $RG = RH'$ for some group $H' \cong H$. Since the last isomorphism immediately yields that $H'^{p^\alpha} \cong H^{p^\alpha}$ and $H'/H'^{p^\alpha} \cong H/H^{p^\alpha}$, it is no harm in assuming that $RG = RH$.

(j) Now, using Lemma 1, we derive $(RG)^{p^\alpha} = R^{p^\alpha}G^{p^\alpha} = R^{p^\alpha}H^{p^\alpha} = (RH)^{p^\alpha}$. Thus $RG^{p^\alpha} = R(R^{p^\alpha}G^{p^\alpha}) = R(R^{p^\alpha}H^{p^\alpha}) = RH^{p^\alpha}$ and we are done.

(jj) Invoking the preceding point we have that $RG^{p^\alpha} = RH^{p^\alpha}$, whence $I(RG^{p^\alpha}; G^{p^\alpha}) = I(RH^{p^\alpha}; H^{p^\alpha})$. Furthermore,

$$I(RG; G^{p^\alpha}) = RG.I(RG^{p^\alpha}; G^{p^\alpha}) = RH.I(RH^{p^\alpha}; H^{p^\alpha}) = I(RH; H^{p^\alpha})$$

and hence

$$R(G/G^{p^\alpha}) \cong RG/I(RG; G^{p^\alpha}) = RH/I(RH; H^{p^\alpha}) \cong R(H/H^{p^\alpha}).$$

□

Referring to [13], for some arbitrary but a fixed positive integer n, a group G is called *$p^{\omega+n}$-projective* if there exists $P \leq G[p^n]$ such that G/P is a direct sum of cyclic groups. Since $G^{p^\omega} \subseteq P$, it is self-evident that $G^{p^{\omega+n}} = 1$, whence length$(G) \leq p^{\omega+n}$. So, if G is $p^{\omega+n}$-projective, then it is $p^{\omega+m}$-projective whenever $m \geq n \in \mathbb{N} \cup \{0\}$. Thus, the direct sums of cyclic groups, being p^ω-projective, are $p^{\omega+n}$-projective for any $n \geq 1$ whereas the converse claim fails; specifically there are many separable $p^{\omega+n}$-projective groups for various naturals n which are not direct sums of cyclic groups. It is also worthwhile noticing that the group G in Theorem D listed above has the property that G/G^{p^ω} is $p^{\omega+1}$-projective. We shall further concentrate on groups G having the property that for some non-negative integer n the factor-group $G/G^{p^{\omega+n}}$ is $p^{\omega+n}$-projective while $G^{p^{\omega+n}}$ may vary.

However, before proceeding, we pause a moment to consider the following simple technical assertions. Specifically, the following hold:

Lemma 3. *Let $K \leq R$ and $A \leq G, B \leq G$. Then*

$$V(RA) \cap V(KB) = V(K(A \cap B)).$$

Proof. It is trivial that the right hand-side is contained in the left hand-side. In order to prove the converse inclusion, take an arbitrary element x in the left hand-side. Thus, $x = \sum_{a \in A} r_a a = \sum_{b \in B} f_b b$ where $r_a \in R$ and $f_b \in K$ and the two sums are in canonical record. Furthermore, $r_a = f_b \in K$ and $a = b \in A \cap B$ which assures that $x \in V(K(A \cap B))$, as required. □

The next statement is our crucial tool (see [4] and [6] too).

Proposition 4. *Let $L \leq R$ be with the same identity such that $R^p \subseteq L$, and $H \leq G$. Then $V(RG)/V(LH)$ is a direct sum of cyclic groups if and only if G/H is a direct sum of cyclic groups.*

Proof. "$\Rightarrow$". It is apparent that $GV(LH)/V(LH) \subseteq V(RG)/V(LH)$, and hence $GV(LH)/V(LH)$ is also a direct sum of cyclic groups (e.g., [8], v. I, p. 110, Theorem 18.1). But it is easily verified that $G \cap V(LH) = H$. Consequently, $GV(LH)/V(LH) \cong G/(G \cap V(LH)) = G/H$, and we are finished.

"$\Leftarrow$". Since G/H is a direct sum of cyclic groups, we may write $G = \cup_{i<\omega} G_i$, where, for each index i, $H \subseteq G_i \subseteq G_{i+1} \leq G$ and $G_i \cap G^{p^i} \subseteq H$ (see, for example, [8], v. I, p. 106, Theorem 17.1). Therefore, $V(RG) = \cup_{i<\omega} V(RG_i)$ and so $V(RG)/V(LH) = \cup_{i<\omega}[V(RG_i)/V(LH)]$. The members of the union obviously form an ascending chain. Moreover, applying Lemmas 1 and 3 and the modular law from [8], we deduce for every index i that

$$(V(RG_i)/V(LH)) \cap (V(RG)/V(LH))^{p^i} =$$

$$(V(RG_i)/V(LH)) \cap (V^{p^i}(RG)V(LH)/V(LH)) =$$
$$(V(RG_i)/V(LH)) \cap (V(R^{p^i}G^{p^i})V(LH)/V(LH)) =$$
$$[V(RG_i) \cap (V(R^{p^i}G^{p^i})V(LH))]/V(LH) =$$
$$V(LH)(V(RG_i) \cap V(R^{p^i}G^{p^i}))/V(LH) =$$
$$V(LH)V(R^{p^i}(G_i \cap G^{p^i}))/V(LH) \subseteq V(LH)/V(LH) = \{0\},$$

which is immediately tantamount to equality.

So, we infer by Kulikov's criterion (see, e.g., [8], v. I, p. 106, Theorem 17.1) that $V(RG)/V(LH)$ is a direct sum of cyclic groups, as asserted. $\square$

As a direct consequence, we yield (see also [4] and [7]):

Corollary 5. *$V(RG)$ is $p^{\omega+n}$-projective if and only if G is $p^{\omega+n}$-projective. In particular, if $RH \cong RG$ as R-algebras and G is $p^{\omega+n}$-projective, then H is $p^{\omega+n}$-projective.*

Proof. The necessity is straightforward since it is readily checked via the aforementioned Nunke's criterion from [13] that subgroups of $p^{\omega+n}$-projective groups are themselves $p^{\omega+n}$-projective.

As for the sufficiency, we write with the aid of Nunke's criterion from [13] that G/P is a direct sum of cyclic groups for some $P \leq G[p^n]$. Observe that $V(RP) \leq V(RG[p^n]) \subseteq V(RG)[p^n]$. Utilizing Proposition 4 we find that $V(RG)/V(RP)$ is also a direct sum of cyclic groups. This allows us to conclude that $V(RG)$ is, in fact, $p^{\omega+n}$-projective.

Finally, it follows at once that $V(RG) \cong V(RH)$ whenever $RG \cong RH$. So, we may apply back-and-forth the preceding equivalence to get the claim. □

A class of groups will be said to be *determined by its p^n-socles*, $n \in \mathbb{N}$, if any two of its members are isomorphic whenever their p^n-socles are isometric (i.e., are isomorphic as valuated groups). It is reasonably clear that the direct sums of cyclic groups are served up to isomorphism by their p-socles. Several important classes of groups are determined by their p^n-socles, where n is a various positive integer. For instance, the most important of them are the following classical ones (e.g., [8], [9] and [10] as well as the bibliography cited there): totally projective groups; $p^{\omega+n}$-projective groups, n is a natural number; direct sums of torsion-complete groups.

Before proceed by proving our main result, we need to recollect the following two remarkable isomorphism claims from ([10], Theorems 3.13 and 3.16). Usually, for any ordinal number α, $f_\alpha(G) = \mathrm{rank}(G^{p^\alpha}[p]/G^{p^{\alpha+1}}[p])$ denotes the α-th Ulm-Kaplansky invariant of G.

Theorem E (Keef, 1994). *Suppose $G/G^{p^{\omega+n}}$ and $H/H^{p^{\omega+n}}$ are $p^{\omega+n}$-projective groups for some $n \in \mathbb{N}$. Then $G \cong H$ if and only if*

(1) $G^{p^{\omega+n}} \cong H^{p^{\omega+n}}$;

(2) $G/G^{p^{\omega+n}} \cong H/H^{p^{\omega+n}}$;

(3) $f_{\omega+n-1}(G) = f_{\omega+n-1}(H)$.

Theorem F (Keef, 1994). *Suppose $n < \omega$ and $\mathcal{K}_n$ is a class of groups determined by its p^n-socles. Let $\mathcal{K}'_n$ be the class of groups such that $G^{p^{\omega+n}} \in \mathcal{K}_n$ and $G/G^{p^{\omega+n}}$ is $p^{\omega+n}$-projective. Then $\mathcal{K}'_n$ is determined by its p^n-socles.*

We now have at our disposal all the machinery necessary to prove the following.

Theorem 6. *Suppose $G \in \mathcal{K}'_1$, where $\mathcal{K}'_1$ is the previously defined class of groups. If $\mathbf{F}_pH \cong \mathbf{F}_pG$ as $\mathbf{F}_p$-algebras for some other group H, then $H \cong G$ and visa versa.*

Proof. The sufficiency is trivial, so we assume that $\mathbf{F}_pG$ and $\mathbf{F}_pH$ are $\mathbf{F}_p$-isomorphic for some group H. In accordance with Proposition 2 we obtain that $\mathbf{F}_pG^{p^{\omega+1}}$ and $\mathbf{F}_pH^{p^{\omega+1}}$ as well as $\mathbf{F}_p(G/G^{p^{\omega+1}})$ and $\mathbf{F}_p(H/H^{p^{\omega+1}})$ are isomorphic over $\mathbf{F}_p$. In view of [1], there is an isometry (i.e., a height preserving group isomorphism) between $G^{p^{\omega+1}}[p]$ and $H^{p^{\omega+1}}[p]$. Moreover, employing Corollary 5, we deduce that $H/H^{p^{\omega+1}}$ must be $p^{\omega+1}$-projective. That is why, since $H^{p^{\omega+1}} \in \mathcal{K}_1$, we infer that $H \in \mathcal{K'}_1$. Again by the usage of [1] we have that $G[p]$ and $H[p]$ are isometric. Henceforth, Theorem F listed above applies to get that G and H are isomorphic, indeed. $\square$

Remark. As aforementioned in the introductory section, $\mathcal{K}_n$ may coincide with the classes of totally projective groups for $n = 1$, $p^{\omega+n}$-projective groups for $n \in \mathbb{N} \cup \{0\}$ and/or (direct sums of) torsion-complete groups for $n = 1$; see [12], [4] and/or [2], [3], [5] respectively.

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