# ONE KANTOROVICH-TYPE OPERATOR

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Abstract. The aim of this paper is to construct a sequence linear positive operators of Kantorovich-type. We demonstrate some convergence and approximation properties of these operators.

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## 1. Introduction

In this section we recall some notions and results which we will use in this paper.

Let $\mathbb{N}$ be the set of positive integer and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m \in \mathbb{N}$, let the operator $K_m : L_1([0,1]) \to C([0,1])$ defined for any function $f \in L_1([0,1])$ by

$$(K_m f)(x) = (m+1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(x)dt, \tag{1}$$

where $p_{m,k}(x)$ are the fundamental polynomials of Bernstein, defined as follows

$$p_{m,k}(x) = \binom{m}{k} x^k (1-x)^{m-k} \tag{2}$$

for any $x \in [0,1]$ and any $k \in \{0, 1, \dots, m\}$.

The operators $K_m$, $m \in \mathbb{N}$ are named Kantorovich operators, introduced and studied in 1930 by L. V. Kantorovich (see [1] or [6]).

In [4] and [5] we give approximation theorems and Voronovskaja-type theorem for these operators.

For $i \in \mathbb{N}_0$, we note

$$(T_{m,i}B_m)(x) = m^i \left(B_m \psi_x^i\right)(x) = m^i \sum_{k=0}^{m} p_{m,k}(x) \left(\frac{k}{m} - x\right)^i$$

where $x \in [0,1]$, $m \in \mathbb{N}$, $\psi_x : [0,1] \to \mathbb{R}$, $\psi_x(t) = t - x$, for any $t \in [0,1]$ and $B_m$, $m \in \mathbb{N}$ are the Bernstein operators.

It is known (see [2]) that

$$(T_{m,0}B_m)(x) = 1, \tag{3}$$

$$(T_{m,1}B_m)(x) = 0, \tag{4}$$

$$(T_{m,2}B_m)(x) = mx(1-x), \tag{5}$$

$$(T_{m,3}B_m)(x) = mx(1-x)(1-2x) \tag{6}$$

and

$$(T_{m,4}B_m)(x) = 3m^2x^2(1-x)^2 + m[x(1-x) - 6x^2(1-x)^2] \tag{7}$$

where $x \in [0,1]$, $m \in \mathbb{N}$.

The following construction and results are given in [4].

We consider $I \subset \mathbb{R}$, $I$ an interval and we shall use the function sets $E(I)$, $F(I)$ which are subsets of the set of real functions defined on $I$, $B(I) = \{f | f : I \to \mathbb{R},\ f \text{ bounded on } I\}$, $C(I) = \{f | f : I \to \mathbb{R},\ f \text{ continuous on } I\}$ and $C_B(I) = B(I) \cap C(I)$. Let $a, b, a', b'$ be real numbers, $a < b$, $a' < b'$, $[a,b] \subset I$, $[a',b'] \subset I$ and $[a,b] \cap [a',b'] \neq \emptyset$.

For $m \in \mathbb{N}$, consider the functions $\varphi_{m,k} : I \to \mathbb{R}$ with the property that $\varphi_{m,k}(x) \geq 0$, for any $x \in [a',b']$, any $k \in \{0,1,\ldots,m]$ and the linear positive functionals $A_{m,k} : E([a,b]) \to \mathbb{R}$, $k \in \{0,1,\ldots,m\}$.

For $m \in \mathbb{N}$, define the operator $L_m : E([a,b]) \to F(I)$ by

$$(L_m f)(x) = \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}(f), \tag{8}$$

for any $f \in E([a,b])$, any $x \in I$ and for $i \in \mathbb{N}_0$, define $T_{m,i}L_m$ by

$$(T_{m,i}L_m)(x) = m^i \left(L_m \psi_x^i\right)(x) = m^i \sum_{k=0}^{m} \varphi_{m,k}(x) A_{m,k}\left(\psi_x^i\right), \tag{9}$$

for any $x \in [a,b] \cap [a',b']$.

In the following, let $s$ be a fixed natural number, $s$ even and we suppose that the operators $(L_m)_{m\geq 1}$ verify the conditions: there exist the smallest $\alpha_s$, $\alpha_{s+2} \in [0,\infty)$ so that

$$\lim_{m\to\infty} \frac{(T_{m,j}L_m)(x)}{m^{\alpha j}} = B_j(x) \in \mathbb{R} \tag{10}$$

for any $x \in [a,b]\cap[a',b']$, $j \in \{s, s+2\}$ and

$$\alpha_{s+2} < \alpha_s + 2. \tag{11}$$

**Theorem 1** *Let $f : [a,b] \to \mathbb{R}$ be a function. If $x \in [a,b]\cap[a',b']$ and $f$ is a $s$ times differentiable function in $x$, the function $f^{(s)}$ is continuous in $x$, then*

$$\lim_{m\to\infty} m^{s-\alpha_s}\left[(L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!}(T_{m,i}L_m)(x)\right] = 0. \tag{12}$$

*If $f$ is a $s$ times differentiable function on $[a,b]$, the function $f^{(s)}$ is continuous on $[a,b]$ and there exist $m(s)\in\mathbb{N}$ and $k_j \in \mathbb{R}$ so that for any $m\in\mathbb{N}$, $m \geq m(s)$ and for any $x\in[a,b]\cap[a',b']$ we have*

$$\frac{(T_{m,j}L_m)(x)}{m^{\alpha_j}} \leq k_j, \tag{13}$$

*where $j\in\{s,s+2\}$, then the convergence given in (12) is uniform on $[a,b]\cap[a',b']$ and*

$$m^{s-\alpha_s}\left|(L_m f)(x) - \sum_{i=0}^{s} \frac{f^{(i)}(x)}{m^i i!}(T_{m,i}L_m)(x)\right| \leq \tag{14}$$
$$\leq \frac{1}{s!}(k_s + k_{s+2})\,\omega\left(f^{(s)}; \frac{1}{\sqrt{m^{2+\alpha_s-\alpha_{s+2}}}}\right),$$

*for any $x\in[a,b]\cap[a',b']$, for any $m\in\mathbb{N}$, $m\geq m(s)$.*

## 2. Preliminaries

**Definition 1** For $m \in \mathbb{N}$, define the operator $\mathcal{K}_m : L_1([0,1]) \to C([0,1])$ by

$$(\mathcal{K}_m f)(x) = m \sum_{k=0}^{m-1} p_{m,k}(x) \int_{\frac{k}{m}}^{\frac{k+1}{m}} f(t)dt + x^m f(1) \tag{15}$$

for any $f \in L_1([0,1])$ and any $x \in [0,1]$.
These operators are Kantorovich-type operators.

**Proposition 1** *The operators $\mathcal{K}_m$, $m \in \mathbb{N}$ are linear and positive on $L_1([0,1])$.*

*Proof.* The proof follows immediately.

**Proposition 2** *For any $m \in \mathbb{N}$ and $x \in [0,1]$, we have*

$$(\mathcal{K}_m e_0)(x) = 1, \tag{16}$$

$$(\mathcal{K}_m e_1)(x) = x + \frac{1}{2m}(1 - x^m), \tag{17}$$

$$(\mathcal{K}_m e_2)(x) = x^2 + \frac{x(2-x)}{m} + \frac{1}{3m^2} - \frac{3m+1}{3m^2} x^m \tag{18}$$

*and*

$$\left(\mathcal{K}_m \psi_x^2\right)(x) = \frac{x(1-x)}{m} + \frac{1}{3m^2} - \frac{3m+1}{3m^2} x^m + \frac{1}{m} x^{m+1}. \tag{19}$$

*Proof.* We have

$$(\mathcal{K}_m e_0)(x) = m \sum_{k=0}^{m-1} p_{m,k}(x) t \Big|_{\frac{k}{m}}^{\frac{k+1}{m}} + x^m = \sum_{k=0}^{m-1} p_{m,k}(x) + x^m =$$

$$= \sum_{k=0}^{m} p_{m,k}(x) = (B_m e_0)(x) = 1,$$

$$\begin{aligned}(\mathcal{K}_m e_1)(x) &= m\sum_{k=0}^{m-1} p_{m,k}(x)\frac{t^2}{2}\Big|_{\frac{k}{m}}^{\frac{k+1}{m}} + x^m = m\sum_{k=0}^{m-1} p_{m,k}(x)\frac{2k+1}{2m^2} + x^m = \\ &= \sum_{k=0}^{m-1} p_{m,k}(x)\,\frac{k}{m} + \frac{1}{2m}\sum_{k=0}^{m-1} p_{m,k}(x) + x^m = \\ &= ((B_m e_1)(x) - p_{m,m}(x)) + \frac{1}{2m}\,((B_m e_0)(x) - p_{m,m}(x)) + x^m,\end{aligned}$$

$$\begin{aligned}(\mathcal{K}_m e_2)(x) &= m\sum_{k=0}^{m-1} p_{m,k}(x)\frac{t^3}{3}\Big|_{\frac{k}{m}}^{\frac{k+1}{m}} + x^m = \\ &= m\sum_{k=0}^{m-1} p_{m,k}(x)\frac{3k^2+3k+1}{3m^3} + x^m = \\ &= ((B_m e_2)(x) - p_{m,m}(x)) + \frac{1}{m}\,((B_m e_1)(x) - p_{m,m}(x)) + \\ &+ \frac{1}{3m^2}\,((B_m e_0)(x) - p_{m,m}(x)) + x^m,\end{aligned}$$

from where the relations (16) - (18) result. From (16) - (18), we obtain the relation (19).

**Remark 1** Taking Proposition 2 into account, from the Theorem Bohman-Korovkin, it results that for any $f \in C([0,1])$ we have $\lim\limits_{m\to\infty} \mathcal{K}_m f = f$ uniform on $[0,1]$.

**Remark 2** From the Theorem Shisha-Mond, approximation theorems for the $(\mathcal{K}_m)_{m\in\mathbb{N}}$ operators result.

## 3. Main result

In the following, we study the $\mathcal{K}_m$, $m \in \mathbb{N}$ operators with the aid of the Theorem 1. For these operators, we have

$$A_{m,k}(f) = \begin{cases} m\displaystyle\int_{\frac{k}{m}}^{\frac{k+1}{m}} f(t)dt, & 0 \le k \le m-1 \\ f(1), & k = m \end{cases} \tag{20}$$

where $m \in \mathbb{N}$ and $f \in L_1([0,1])$.

**Theorem 2** *For $m, i \in \mathbb{N}_0$, $m \neq 0$ and $x \in [0,1]$ we have*

$$(T_{m,i}\mathcal{K}_m)(x) = \frac{1}{i+1}\sum_{j=0}^{i}\binom{i+1}{j}(T_{m,j}B_m)(x) - \tag{21}$$
$$-\frac{x^m}{i+1}\left[(1+m(1-x))^{i+1} - (m(1-x))^{i+1}\right] + x^m(m(1-x))^i.$$

*Proof.* Taking (9) and (20) into account, we have

$$(T_{m,i}\mathcal{K}_m)(x) = m^i\left(\mathcal{K}_m\psi_x^i\right)(x) =$$
$$= m^i\left[m\sum_{k=0}^{m-1}p_{m,k}(x)\int_{\frac{k}{m}}^{\frac{k+1}{m}}(t-x)^i dt + x^m(1-x)^i\right] =$$
$$= m^i\left[m\sum_{k=0}^{m-1}p_{m,k}(x)\frac{(t-x)^{i+1}}{i+1}\Bigg|_{\frac{k}{m}}^{\frac{k+1}{m}} + x^m(1-x)^i\right] =$$
$$= m^i\left[\frac{m}{i+1}\sum_{k=0}^{m-1}p_{m,k}(x)\left(\left(\frac{k}{m}-x+\frac{1}{m}\right)^{i+1}-\left(\frac{k}{m}-x\right)^{i+1}\right)+x^m(1-x)^i\right] =$$
$$= m^i\left[\frac{m}{i+1}\sum_{k=0}^{m-1}p_{m,k}(x)\sum_{j=0}^{i}\binom{i+1}{j}\left(\frac{k}{m}-x\right)^j\left(\frac{1}{m}\right)^{i+1-j} + x^m(1-x)^i\right] =$$
$$= m^i\left\{\frac{m}{i+1}\sum_{j=0}^{i}\binom{i+1}{j}\frac{1}{m^{i+1-j}}\left[\sum_{k=0}^{m}p_{m,k}(x)\left(\frac{k}{m}-x\right)^j - p_{m,m}(x)(1-x)^j\right] +\right.$$
$$\left. + x^m(1-x)^i\right\} = \frac{1}{i+1}\sum_{j=0}^{i}\binom{i+1}{j}\left[(T_{m,j}B_m)(x) - x^m(m(1-x))^j\right] +$$
$$+ x^m(m(1-x))^i = \frac{1}{i+1}\sum_{j=0}^{i}\binom{i+1}{j}(T_{m,j}B_m)(x) -$$
$$-\frac{x^m}{i+1}\sum_{j=0}^{i}\binom{i+1}{j}(m(1-x))^j + x^m(m(1-x))^i,$$

from where we obtain relation (21).

**Remark 3** If $m, i \in \mathbb{N}$ and $x \in [0,1]$, from (21) it results that

$$(T_{m,i}\mathcal{K}_m)(x) = \tag{22}$$
$$= \frac{1}{i+1}\sum_{j=0}^{i}\binom{i+1}{j}(T_{m,j}B_m)(x) - \frac{x^m}{i+1}\sum_{j=0}^{i-1}\binom{i+1}{j}(m(1-x))^j.$$

**Lemma 1** *For any $m \in \mathbb{N}$ and $x \in [0,1]$, we have*

$$(T_{m,0}\mathcal{K}_m)(x) = 1, \tag{23}$$

$$(T_{m,1}\mathcal{K}_m)(x) = \frac{1}{2}(1-x^m), \tag{24}$$

$$(T_{m,2}\mathcal{K}_m)(x) = \frac{1}{3}(1-x^m) + mx(1-x)(1-x^{m-1}) \tag{25}$$

*and*

$$(T_{m,4}\mathcal{K}_m)(x) = \frac{1}{5}(1-x^m) + mx(1-x)(6x^2+2x+5-x^{m-1}) + \tag{26}$$
$$+ m^2x^2(1-x)^2(3-2x^{m-2}) - 2m^3x^m(1-x)^3.$$

*Proof.* It results from the Theorem 2 and relations (3) - (7).

**Lemma 2** *We have*

$$\lim_{m\to\infty}(T_{m,0}\mathcal{K}_m)(x) = 1, \tag{27}$$

$$\lim_{m\to\infty}\frac{(T_{m,2}\mathcal{K}_m)(x)}{m} = x(1-x), \tag{28}$$

$$\lim_{m\to\infty}\frac{(T_{m,4}\mathcal{K}_m)(x)}{m^2} = 3x^2(1-x)^2 \tag{29}$$

*for any $x \in [0,1]$ and*

$$(T_{m,0}\mathcal{K}_m)(x) = 1 = k_0, \tag{30}$$

$$\frac{(T_{m,2}\mathcal{K}_m)(x)}{m} \le \frac{7}{12} = k_2 \tag{31}$$

*and*

$$\frac{(T_{m,4}\mathcal{K}_m)(x)}{m^2} \le \frac{291}{80} = k_4 \tag{32}$$

*for any $x \in [0,1]$, any $m \in \mathbb{N}$.*

*Proof.* For $x = 1$, the relations (28) and (29) hold. For $x \in [0,1)$ we take $\lim\limits_{m\to\infty} x^m = 0$, $\lim\limits_{m\to\infty} mx^m = 0$ into account, so (28) and (29) hold.

We have

$$\frac{(T_{m,2}\mathcal{K}_m)(x)}{m} = \frac{1}{3m}(1-x^m) + x(1-x)(1-x^{m-1}) \leq$$
$$\leq \frac{1}{3m} + x(1-x) \leq \frac{1}{3} + \frac{1}{4} = \frac{7}{12},$$

because $x(1-x) \leq \dfrac{1}{4}$, for any $x \in [0,1]$ and with similar calculation we obtain the inequality (32).

**Theorem 3** *Let $f : [0,1] \to \mathbb{R}$ be a function. If $f$ is a continuous function in $x \in [0,1]$, then*

$$\lim_{m\to\infty} (\mathcal{K}_m f)(x) = f(x). \tag{33}$$

*If $f$ is continuous on $[0,1]$, then the convergence given in (33) is uniform on $[0,1]$ and*

$$|(\mathcal{K}_m f)(x) - f(x)| \leq \frac{19}{12}\,\omega\left(f; \frac{1}{\sqrt{m}}\right) \tag{34}$$

*for any $x \in [0,1]$, any $m \in \mathbb{N}$.*

*Proof.* It results from Theorem 1 for $s = 0$, Lemma 1 and Lemma 2.

**Theorem 4** *Let $f : [0,1] \to \mathbb{R}$ be a function. If $x \in [0,1]$ and $f$ is two times differentiable function in $x$, the function $f^{(2)}$ is continuous in $x$, then*

$$\lim_{m\to\infty} m\left[(\mathcal{K}_m f)(x) - f(x)\right] = \frac{1}{2} f^{(1)}(x) + \frac{1}{2}\,x(1-x) f^{(2)}(x). \tag{35}$$

*If $f$ is a two times differentiable function on $[0,1]$, the function $f^{(2)}$ is continuous on $[0,1]$, then the convergence given in (35) is uniform on $[0,1]$ and*

$$m\Big|(\mathcal{K}_m f)(x) - f(x) - \frac{1}{2m}(1-x^m) f^{(1)}(x) - \tag{36}$$
$$- \frac{1}{2m^2}\left[\frac{1}{3}(1-x^m) + mx(1-x)(1-x^{m-1})\right] f^{(2)}(x)\Big| \leq$$
$$\leq \frac{1013}{480}\,\omega\left(f^{(2)}; \frac{1}{\sqrt{m}}\right)$$

*for any $x \in [0,1]$, any $m \in \mathbb{N}$.*

*Proof.* It results from Theorem 1 for $s = 2$, Lemma 1 and Lemma 2.

**Remark 4** The relation (35) is a Voronovskaja-type relation for the $(\mathcal{K}_m)_{m\geq 1}$ operators.

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