ON FIXED POINTS OF PSEUDOCONTRACTIVE MAPPINGS

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Abstract. Suppose $E = L_p$ (or l_p), $p \geq 2$, and C is a nonempty closed convex subset of E. Let $T : C \to C$ be a continuous pseudocontractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences, satisfying the following conditions:

(i) $0 \leq \alpha_n, \beta_n, \delta_n \leq 1$, $0 < \gamma_n < 1$;

(ii) $\alpha_n + \beta_n + \gamma_n + \delta_n = 1$;

(iii) $\lim\limits_{n\to\infty} \beta_n = 0 = \lim\limits_{n\to\infty} \alpha_n$;

(iv) $\sum\limits_{n=0}^{\infty} \dfrac{\alpha_n}{\alpha_n + \beta_n + \delta_n} = \infty$;

(v) $\delta_n = o(\alpha_n)$.

For arbitrary initial value $x_1 \in C$ and a fixed anchor $u \in C$, the sequence $\{x_n\}$ is defined by $x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T x_n + \delta_n u_n, n \geq 1$, where $\{u_n\}$ is abounded sequence of error terms. Then $\{x_n\}$ converges strongly to a fixed point of T.

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1. Introduction

Let E be a real Banach space and E^* be its dual space. The normalized duality mapping $J : E \to 2^{E^*}$ is defined as

$$J(x) := \left\{ x^* \in E^*; \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}.$$

Let C a closed convex subset of E. The mapping $T : C \to C$ is called pseudocontractive if

$$\|x - y\| \le \|x - y + t\left((I - T)x - (I - T)y\right)\|,$$

holds for every $x, y \in C$ and $t > 0$. An equivalent definition of pseudocontractive mappings is due to Kato [3],

$$\langle Tx - Ty, j(x - y)\rangle \le \|x - y\|^2,$$

for $x, y \in C$ and $j(x - y) \in J(x - y)$.

Let $U = \{x \in E : \|x\| = 1\}$ denote the unit sphere of E. The norm on E is said to be Gateaux differentiable if the

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}, \tag{1}$$

exist for each $x, y \in U$ and in this case E is said to be smooth. E is said to have a uniformly Frechet differentiable norm if the limit (1) is attained uniformly for $x, y \in U$ and in this case E is said to be uniformly smooth. It is well known that if E is uniformly smooth then the duality mapping is norm-to-norm uniformly continuous on bounded subset of E.

Very recently, Yao et al. [5], introduced the following iterative scheme: Let C be a closed convex subset of real Banach space E and $T : C \to C$ be a mapping. Define $\{x_n\}$ in the following way:

$$\begin{aligned} &x_1 \in C, \\ &x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T x_n,\ n \ge 1, \end{aligned} \tag{2}$$

where u is an anchor and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are three real sequences in $(0, 1)$ satisfying some appropriate conditions.

The following theorem is due to Yao et al. [5].

Theorem 1. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space E. Let $T : C \to C$ be a continuous pseudocontractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be three real sequences in $(0, 1)$ satisfying the following conditions:*
(i) $\alpha_n + \beta_n + \gamma_n = 1$;
(ii) $\lim_{n\to\infty} \beta_n = 0$ and $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$;

(iii) $\sum_{n=0}^{\infty} \frac{\alpha_n}{\alpha_n + \beta_n} = \infty$.

For arbitrary initial value $x_1 \in C$ and fixed anchor $u \in C$, the sequence $\{x_n\}$ is defined by (2). Then $\{x_n\}$ converges strongly to a fixed point of T.

Suppose now $E = L_p$, (or l_p), $p \geq 2$, $C \subset E$ and j will always denote the single-valued normalized duality mapping of E into E^*.

In this paper, we modified the results of Yao et al. [5] for the implicit Mann type iteration process with errors, associated with pseudocontractive mappings to have the strong convergence in the setting of L_p (or l_p), $p \geq 2$ spaces.

We shall need the following results.

Lemma 1. *[2] For the Banach space $E = L_p$, (or l_p), $p \geq 2$, the following inequality holds for all x, y in E:*

$$\|x+y\|^2 \leq (p-1)\|x\|^2 + \|y\|^2 + 2\langle x, j(y)\rangle.$$

Lemma 2. *[4] Let β_n be a nonnegative sequence satisfying*

$$\beta_{n+1} \leq (1-\delta_n)\beta_n + \sigma_n,$$

with $\delta_n \in [0,1]$, $\sum_{i=1}^{\infty} \delta_i = \infty$, and $\sigma_n = o(\delta_n)$. Then $\lim_{n\to\infty} \beta_n = 0$.

2. Main results

Now we prove our main results.

Theorem 2. *Suppose $E = L_p$ (or l_p), $p \geq 2$, and C is a nonempty closed convex subset of E. Let $T : C \to C$ be a continuous pseudocontractive mapping. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ be four real sequences, satisfying the following conditions:*

$$\begin{array}{ll}
(i) & 0 \leq \alpha_n, \beta_n, \delta_n \leq 1,\ 0 < \gamma_n < 1; \\
(ii) & \alpha_n + \beta_n + \gamma_n + \delta_n = 1; \\
(iii) & \lim_{n\to\infty} \beta_n = 0 = \lim_{n\to\infty} \alpha_n; \\
(iv) & \sum_{n=0}^{\infty} \frac{\alpha_n}{\alpha_n + \beta_n + \delta_n} = \infty; \\
(v) & \delta_n = o(\alpha_n).
\end{array}$$

For arbitrary initial value $x_1 \in C$ and a fixed anchor $u \in C$, the sequence $\{x_n\}$ is defined by

$$x_1 \in C,$$
$$x_n = \alpha_n u + \beta_n x_{n-1} + \gamma_n T x_n + \delta_n u_n, n \geq 1, \tag{3}$$

where $\{u_n\}$ is abounded sequence of error terms. Then $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Indeed, suppose we take a fixed point x^* of T. Since $\{u_n\}$ is a bounded sequence of error terms, set $M_1 = \sup_{n\geq 1} \|u_n - x^*\|$. First, we show that $\{x_n\}$ is bounded. Consider

$$\begin{aligned} x_n - x^* &= (1-\gamma_n)\left(\frac{\alpha_n}{1-\gamma_n}u + \frac{\beta_n}{1-\gamma_n}x_{n-1} + \frac{\delta_n}{1-\gamma_n}u_n\right) + \gamma_n T x_n - x^* \\ &= (1-\gamma_n)\left[\frac{\alpha_n}{1-\gamma_n}(u-x^*) + \frac{\beta_n}{1-\gamma_n}(x_{n-1}-x^*)\right. \\ &+ \left.\frac{\delta_n}{1-\gamma_n}(u_n-x^*)\right] + \gamma_n(Tx_n - x^*). \end{aligned}$$

It follows that

$$\begin{aligned} \|x_n - x^*\|^2 &= \langle x_n - x^*, j(x_n - x^*)\rangle \\ &= \left\langle (1-\gamma_n)\left[\frac{\alpha_n}{1-\gamma_n}(u-x^*) + \frac{\beta_n}{1-\gamma_n}(x_{n-1}-x^*)\right.\right. \\ &+ \left.\left.\frac{\delta_n}{1-\gamma_n}(u_n-x^*)\right] + \gamma_n(Tx_n - x^*), j(x_n - x^*)\right\rangle \\ &= (1-\gamma_n)\left\langle \frac{\alpha_n}{1-\gamma_n}(u-x^*) + \frac{\beta_n}{1-\gamma_n}(x_{n-1}-x^*)\right. \\ &+ \left.\frac{\delta_n}{1-\gamma_n}(u_n-x^*), j(x_n - x^*)\right\rangle + \gamma_n\langle Tx_n - x^*, j(x - x^*)\rangle \\ &\leq (1-\gamma_n)\left\|\frac{\alpha_n}{1-\gamma_n}(u-x^*) + \frac{\beta_n}{1-\gamma_n}(x_{n-1}-x^*)\right. \\ &+ \left.\frac{\delta_n}{1-\gamma_n}(u_n-x^*)\right\| \|x_n - x^*\| + \gamma_n \|x_n - x^*\|^2, \end{aligned}$$

implies

$$\begin{aligned}
\|x_n-x^*\| &\le \left\|\frac{\alpha_n}{1-\gamma_n}(u-x^*)+\frac{\beta_n}{1-\gamma_n}(x_{n-1}-x^*)+\frac{\delta_n}{1-\gamma_n}(u_n-x^*)\right\| \qquad (4)\\
&\le \frac{\alpha_n}{1-\gamma_n}\|u-x^*\|+\frac{\beta_n}{1-\gamma_n}\|x_{n-1}-x^*\|+\frac{\delta_n}{1-\gamma_n}\|u_n-x^*\|\\
&\le \frac{\alpha_n}{1-\gamma_n}\|u-x^*\|+\frac{\beta_n}{1-\gamma_n}\|x_{n-1}-x^*\|+M_1\frac{\delta_n}{1-\gamma_n}\\
&\le \max\{\|u-x^*\|,\|x_{n-1}-x^*\|,M_1\}.
\end{aligned}$$

Now, induction yields

$$\|x_n-x^*\|\le \max\{\|u-x^*\|,\|x_1-x^*\|,M_1\},$$

implies $\{x_n\}$ is bounded and so is $\{Tx_n\}$. Let

$$M=\sup_{n\ge1}\|x_n-x^*\|+\sup_{n\ge1}\|Tx_n-x^*\|+M_1.$$

Finally, we prove that $x_n\to x^*$. Since $\delta_n=o(\alpha_n)$, implies there exist a sequence $\{t_n\}$ such that $t_n\to 0$ as $n\to\infty$ and $\delta_n=t_n\alpha_n$. Now

$$\begin{aligned}
&\left\|\frac{\alpha_n}{1-\gamma_n}(u-x^*)+\frac{\delta_n}{1-\gamma_n}(u_n-x^*)\right\| \le \frac{\alpha_n}{1-\gamma_n}\|u-x^*\|+\frac{\delta_n}{1-\gamma_n}\|u_n-x^*\| \qquad (5)\\
&\le \frac{\alpha_n}{1-\gamma_n}\|u-x^*\|+M\frac{\delta_n}{1-\gamma_n}=\frac{\alpha_n}{1-\gamma_n}(\|u-x^*\|+Mt_n).
\end{aligned}$$

From Lemma 1 and relations (4), (5), we have

$$
\begin{aligned}
\|x_n - x^*\|^2 &= \left\| \frac{\alpha_n}{1-\gamma_n}(u-x^*) + \frac{\beta_n}{1-\gamma_n}(x_{n-1}-x^*) + \frac{\delta_n}{1-\gamma_n}(u_n - x^*) \right\|^2 \\
&\le \left(\frac{\beta_n}{1-\gamma_n}\right)^2 \|x_{n-1}-x^*\|^2 + (p-1)\left\| \frac{\alpha_n}{1-\gamma_n}(u-x^*) + \frac{\delta_n}{1-\gamma_n}(u_n-x^*)\right\|^2 \\
&+ 2\left\langle \frac{\alpha_n}{1-\gamma_n}(u-x^*) + \frac{\delta_n}{1-\gamma_n}(u_n-x^*), j\left(\frac{\beta_n}{1-\gamma_n}(x_{n-1}-x^*)\right)\right\rangle \\
&\le \left(1-\frac{\alpha_n}{1-\gamma_n}\right)\|x_{n-1}-x^*\|^2 + (p-1)\left\|\frac{\alpha_n}{1-\gamma_n}(u-x^*)\right. \\
&+ \left.\frac{\delta_n}{1-\gamma_n}(u_n-x^*)\right\|^2 \\
&+ 2\frac{\beta_n}{1-\gamma_n}\left\|\frac{\alpha_n}{1-\gamma_n}(u-x^*) + \frac{\delta_n}{1-\gamma_n}(u_n-x^*)\right\| \|x_{n-1}-x^*\| \\
&\le \left(1-\frac{\alpha_n}{1-\gamma_n}\right)\|x_{n-1}-x^*\|^2 + (p-1)\left(\frac{\alpha_n}{1-\gamma_n}\right)^2 (\|u-x^*\| + Mt_n)^2 \\
&+ 2M\frac{\alpha_n\beta_n}{(1-\gamma_n)^2}(\|u-x^*\| + Mt_n) \\
&= \left(1-\frac{\alpha_n}{1-\gamma_n}\right)\|x_{n-1}-x^*\|^2 + \frac{\alpha_n}{1-\gamma_n}\eta_n,
\end{aligned}
$$

where

$$
\eta_n = \left[(p-1)\frac{\alpha_n}{1-\gamma_n}(\|u-x^*\| + Mt_n) + 2M\frac{\beta_n}{1-\gamma_n}\right](\|u-x^*\| + Mt_n).
$$

Now according to Lemma 2, we have $x_n \to x^*$.

Remark 1. *Our results are true for L_p (or l_p), $p \geq 2$ space (Banach spaces) instead of uniformly smooth Banach spaces.*

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