THE INTEGRAL OPERATOR ON THE $SP(\alpha, \beta)$ CLASS

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Abstract. In this paper we present a convexity condition for a integral operator F defined in formula (2) on the class $SP(\alpha, \beta)$.

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1.INTRODUCTION

Let $U = \{z \in C, |z| < 1\}$ be the unit disc of the complex plane and denote by $H(U)$, the class of the olomorphic functions in U. Consider $A = \{f \in H(U), f(z) = z + a_2z^2 + a_3z^3 + ..., z \in U\}$ be the class of analytic functions in U and $S = \{f \in A : f \text{ is univalent in } U\}$.

Denote with K the class of the olomorphic functions in U with $f(0) = f'(0) - 1 = 0$, where is convex functions in U , defined by

$$K = \left\{ f \in H(U) : f(0) = f'(0) - 1 = 0, \mathbf{Re}\left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > 0, z \in U \right\}$$

and denote for $K(\mu)$ the functions convex by the order μ, $\mu \in [0, 1)$, defined by

$$K(\mu) = \left\{ f \in H(U) : f(0) = f'(0) - 1 = 0, \mathbf{Re}\left\{ \frac{zf''(z)}{f'(z)} + 1 \right\} > \mu, z \in U \right\}.$$

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In the paper (3) F. Ronning introduced the class of univalent functions $SP(\alpha,\beta), \alpha > 0, \beta \in [0,1)$, the class of all functions $f \in S$ which have the property:

$$\left|\frac{zf'(z)}{f(z)} - (\alpha+\beta)\right| \leq \mathbf{Re}\frac{zf'(z)}{f(z)} + \alpha - \beta, \tag{1}$$

for all $z \in U$.

Geometric interpretation: $f \in SP(\alpha,\beta)$ if and only if $zf'(z)/f(z), z \in U$ takes all values in the parabolic region

$$\Omega_{\alpha,\beta} = \{\omega : |\omega - (\alpha+\beta)| \leq \mathbf{Re}\omega + \alpha - \beta\} = \left\{\omega = u + iv : v^2 \leq 4\alpha(u-\beta)\right\}.$$

We consider the integral operator defined in [2]

$$F(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(t)}{t}\right)^{\alpha_n} dt \tag{2}$$

and we study your properties.

Remark. We observe that for $n = 1$ and $\alpha_1 = 1$ we obtain the integral operator of Alexander.

2.Main results

Theorem 1.*Let $\alpha_i, i \in \{1,...,n\}$ the real numbers with the properties $\alpha_i > 0$ for $i \in \{1,...,n\}$,*

$$\sum_{i=1}^{n} \alpha_i < \frac{1}{\alpha - \beta + 1} \tag{3}$$

and $(\beta - \alpha - 1)\sum_{i=1}^{n}\alpha_i + 1 \in (0,1)$. We suppose that the functions $f_i \in SP(\alpha,\beta)$ for $i = \{1,...,n\}$ and $\alpha > 0, \beta \in [0,1)$. In this conditions the integral operator defined in (2) is convex by the order $(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i + 1$.

Proof. We calculate for F the derivatives of the first and second order.
From (2) we obtain:

$$F'(z) = \left(\frac{f_1(z)}{z}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{f_n(z)}{z}\right)^{\alpha_n}$$

and

$$F''(z) = \sum_{i=1}^{n} \alpha_i \left(\frac{f_i(z)}{z}\right)^{\alpha_i - 1} \left(\frac{z f_i'(z) - f_i(z)}{z f_i(z)}\right) \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{f_j(z)}{z}\right)^{\alpha_j}$$

$$\frac{F''(z)}{F'(z)} = \alpha_1 \left(\frac{z f_1'(z) - f_1(z)}{z f_1(z)}\right) + \ldots + \alpha_n \left(\frac{z f_n'(z) - f_n(z)}{z f_n(z)}\right).$$

$$\frac{F''(z)}{F'(z)} = \alpha_1 \left(\frac{f_1'(z)}{f_1(z)} - \frac{1}{z}\right) + \ldots + \alpha_n \left(\frac{f_n'(z)}{f_n(z)} - \frac{1}{z}\right). \tag{4}$$

Multiply the relation (4) with z we obtain:

$$\frac{zF''(z)}{F'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{z f_i'(z)}{f_i(z)} - 1\right) = \sum_{i=1}^{n} \alpha_i \frac{z f_i'(z)}{f_i(z)} - \sum_{i=1}^{n} \alpha_i. \tag{5}$$

The relation (5) is equivalent with

$$\frac{zF''(z)}{F'(z)} = \sum_{i=1}^{n} \alpha_i \left(\frac{z f_i'(z)}{f_i(z)} + \alpha - \beta\right) + (\beta - \alpha - 1) \sum_{i=1}^{n} \alpha_i. \tag{6}$$

and

$$\frac{zF''(z)}{F'(z)} + 1 = \sum_{i=1}^{n} \alpha_i \left(\frac{z f_i'(z)}{f_i(z)} + \alpha - \beta\right) + (\beta - \alpha - 1) \sum_{i=1}^{n} \alpha_i + 1. \tag{7}$$

We calculate the real part from both terms of the above equality and obtain:

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)} + 1\right) = \mathbf{Re}\left\{\sum_{i=1}^{n} \alpha_i \left(\frac{z f_i'(z)}{f_i(z)} + \alpha - \beta\right)\right\} + (\beta - \alpha - 1) \sum_{i=1}^{n} \alpha_i + 1. \tag{8}$$

Because $f_i \in SP(\alpha,\beta)$ for $i = \{1,...,n\}$ we apply in the above relation the inequality (1) and obtain:

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right) \geq \sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-(\alpha+\beta)\right|+(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i+1. \quad (9)$$

Because $\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-(\alpha+\beta)\right| > 0$ for all $i \in \{1,...,n\}$ and the inequality (3) we obtain that

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right) \geq (\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i+1 > 0. \quad (10)$$

From (10) and because $(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i+1 \in (0,1)$ we obatin that the integral operator defined in (2) is convex by the order $(\beta-\alpha-1)\sum_{i=1}^{n}\alpha_i+1$.

Corollary 2. *Let γ the real numbers with the properties $0 < \gamma < \frac{1}{\alpha-\beta+1}$. We suppose that the functions $f \in SP(\alpha,\beta)$ and $\alpha > 0, \beta \in [0,1)$. In this conditions the integral operator $F(z) = \int_0^z \left(\frac{f(z)}{z}\right)^{\gamma} dt$ is convex.*

Proof. In the Theorem 1, we consider $n = 1$, $\alpha_1 = \gamma$ and $f_1 = f$.

For $\alpha = \beta \in (0,1)$ we obtain the class $S(\alpha,\alpha)$ where is caracterized by the next property

$$\left|\frac{zf'(z)}{f(z)}-2\alpha\right| \leq \mathbf{Re}\frac{zf'(z)}{f(z)}. \quad (11)$$

Corollary 3. *Let $\alpha_i, i \in \{1,...,n\}$ the real numbers with the properties $\alpha_i > 0$ for $i \in \{1,...,n\}$ and*

$$1-\sum_{i=1}^{n}\alpha_i \in [0,1). \quad (12)$$

We suppose that the functions $f_i \in SP(\alpha,\alpha)$ for $i = \{1,...,n\}$ and $\alpha \in (0,1)$. In this conditions the integral operator defined in (2) is convex by the order $1-\sum_{i=1}^{n}\alpha_i$.

Proof. From (2) obtain that

$$\frac{zF''(z)}{F'(z)}=\sum_{i=1}^{n}\alpha_i\frac{zf_i'(z)}{f_i(z)}-\sum_{i=1}^{n}\alpha_i, \tag{13}$$

where is equivalent with

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right)=\sum_{i=1}^{n}\alpha_i\mathbf{Re}\frac{zf_i'(z)}{f_i(z)}-\sum_{i=1}^{n}\alpha_i+1, \tag{14}$$

From (11) and (14) obtain that:

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right)>\sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-2\alpha\right|+1-\sum_{i=1}^{n}\alpha_i. \tag{15}$$

Because $\sum_{i=1}^{n}\alpha_i\left|\frac{zf_i'(z)}{f_i(z)}-2\alpha\right|>0$ for all $i\in\{1,...,n\}$, from (15) we obatin:

$$\mathbf{Re}\left(\frac{zF''(z)}{F'(z)}+1\right)>1-\sum_{i=1}^{n}\alpha_i. \tag{16}$$

Now, from (12) we obtain that the operator defined in (2) is convex by the order $1-\sum_{i=1}^{n}\alpha_i$.

For $\alpha=\beta=\frac{1}{2}$ we observe that $SP\left(\frac{1}{2},\frac{1}{2}\right)=SP$. This class is defined by Ronning in the paper [4]. For the class SP we have the next result:

Corollary 4.*Let* $\alpha_i, i\in\{1,...,n\}$ *the real numbers with the properties* $\alpha_i>0$ *for* $i\in\{1,...,n\}$ *and*

$$1-\sum_{i=1}^{n}\alpha_i\in[0,1). \tag{17}$$

We suppose that the functions $f_i\in SP$ *for* $i=\{1,...,n\}$*. In this conditions the integral operator defined in (2) is convex by the order* $1-\sum_{i=1}^{n}\alpha_i$*.*

References

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