A NEW INTEGRAL UNIVALENT OPERATOR

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Abstract. In this paper the authors introduced an integral operator and proved its properties.

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1.Introduction

Let $U = \{z \in C, |z| < 1\}$ be the unit disc of the complex plane and denote by $H(U)$, the class of the olomorphic functions in U. Consider $A = \{f \in H(U), f(z) = z + a_2z^2 + a_3z^3 + ..., z \in U\}$ be the class of analytic functions in U and $S = \{f \in A : f \text{ is univalent in } U\}$.

Consider S^*, the class of starlike functions in unit disk, defined by

$$S^* = \left\{ f \in H(U) : f(0) = f'(0) - 1 = 0, \mathbf{Re}\frac{zf'(z)}{f(z)} > 0, z \in U \right\}.$$

A function $f \in S$ is the starlike function of order $\alpha, 0 \leq \alpha < 1$ and denote this class by $S^*(\alpha)$ if f verify the inequality

$$\mathbf{Re}\frac{zf'(z)}{f(z)} > \alpha, z \in U.$$

Denote with K the class of convex functions in U, defined by

$$K = \left\{ f \in H(U) : f(0) = f'(0) - 1 = 0, \mathbf{Re}\left\{\frac{zf''(z)}{f'(z)} + 1\right\} > 0, z \in U \right\}.$$

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A function $f \in S$ is convex function of order $\alpha, 0 \leq \alpha < 1$ and denote this class by $K(\alpha)$ if f verify the inequality

$$\mathbf{Re}\left\{\frac{zf''(z)}{f'(z)}+1\right\} > \alpha, z \in U.$$

A function $f \in UCV$ if and only if

$$\mathbf{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, z \in U.$$

S. N. Kudryashov in 1973 investigated the maximum value of M such that the inequality

$$\left|\frac{f''(z)}{f'(z)}\right| \leq M$$

implies that f is univalent in U. He showed that if $M = 3,05...$ and

$$\left|\frac{f''(z)}{f'(z)}\right| \leq 3,05...$$

where M is the solution of the equation $8\left[M(M-2)^3\right]^{1/2} - 3(3-M)^2 = 12$, then f is univalent in U.

Also, if

$$\left|\frac{f''(z)}{f'(z)}\right| \leq 2,8329...,$$

then the function f is starlike in unit disk. This result is obtained by Miller and Mocanu.

Consider the new general integral operator defined by the formula:

$$F_{\alpha_1,...,\alpha_n}(z) = \int_0^z (f_1'(t))^{\alpha_1} ... (f_n'(t))^{\alpha_n}\, dt. \tag{1}$$

2.Main results

In the next theorem we proved the univalence for this operator.

Theorem 1. *Let $\alpha_i \in \mathbf{R}, i \in \{1,...,n\}$ and $\alpha_i > 0$. Consider $f_i \in S$ (univalent functions) and suppose that*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq M, \tag{2}$$

where $M = 3,05...$, *for all* $z \in U$.

If $\sum_{i=1}^{n} \alpha_i \leq 1$ *the integral operator* $F_{\alpha_1,...,\alpha_n}$ *is univalent.*

Proof. We have

$$\frac{F''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)} = \alpha_1 \frac{f_1''(z)}{f_1'(z)} + ... + \alpha_n \frac{f_n''(z)}{f_n'(z)}. \tag{3}$$

that is equivalent with,

$$\left|\frac{F''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)}\right| \leq \alpha_1 \left|\frac{f_1''(z)}{f_1'(z)}\right| + ... + \alpha_n \left|\frac{f_n''(z)}{f_n'(z)}\right|.$$

Now, applying the inequality (2), we obtain that

$$\left|\frac{F''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)}\right| \leq \alpha_1 M + ... + \alpha_n M = M \sum_{i=1}^{n} \alpha_i \leq M.$$

The last inequality implies that the integral operator $F_{\alpha_1,...,\alpha_n}$ is univalent.

Theorem 2. *Let* $\alpha_i \in \mathbf{R}, i \in \{1,...,n\}$ *and* $\alpha_i > 0$. *Consider* $f_i \in S$ *(univalent functions) and suppose that*

$$\left|\frac{f_i''(z)}{f_i'(z)}\right| \leq M_1, \tag{4}$$

where $M_1 = 2,8329...$, *is the smallest root of equation* $x sin x + cos x = 1/e$.

If $\sum_{i=1}^{n} \alpha_i \leq 1$, *the integral operator* $F_{\alpha_1,...,\alpha_n}$ *is starlike.*

Proof. We have

$$\frac{F''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)} = \alpha_1 \frac{f_1''(z)}{f_1'(z)} + ... + \alpha_n \frac{f_n''(z)}{f_n'(z)}. \tag{5}$$

that is equivalent with,

$$\left|\frac{F''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)}\right| \leq \alpha_1 \left|\frac{f''_1(z)}{f'_1(z)}\right| + ... + \alpha_n \left|\frac{f''_n(z)}{f'_n(z)}\right|.$$

Now, applying the inequality (4), we obtain that

$$\left|\frac{F''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)}\right| \leq \alpha_1 M_1 + ... + \alpha_n M_1 = M_1 \sum_{i=1}^{n} \alpha_i \leq M_1.$$

The last inequality implies that the integral operator $F_{\alpha_1,...,\alpha_n}$ is starlike.

Theorem 3. *Let $\alpha_i \in \mathbf{R}, i \in \{1,...,n\}$ and $\alpha_i > 0$. Consider $f_i \in K$ (convex functions), for all $i \in \{1,...,n\}$. Then the integral operator $F_{\alpha_1,...,\alpha_n}$ is convex.*

Proof.

$$\mathbf{Re}\left\{\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)} + 1\right\} = \sum_{i=1}^{n} \alpha_i \mathbf{Re}\left(\frac{zf''_i(z)}{f'_i(z)} + 1\right) \geq 0.$$

Since

$$\mathbf{Re}\left(\frac{zf''_i(z)}{f'_i(z)} + 1\right) \geq 0$$

for all $i \in \{1,...,n\}$, we have the integral operator $F_{\alpha_1,...,\alpha_n}$ is convex.

Theorem 4. *Let $\alpha_i \in \mathbf{R}, i \in \{1,...,n\}$ and $\alpha_i > 0$. Consider $f_i \in K(\beta_i)$, $0 \leq \beta_i < 1$, for all $i \in \{1,...,n\}$. In these conditions, the integral operator $F_{\alpha_1,...,\alpha_n}$ is convex of order $\sum_{i=1}^{n} \alpha_i(\beta_i - 1) + 1$, where $0 \leq \sum_{i=1}^{n} \alpha_i(\beta_i - 1) + 1 < 1$.*

Proof. We have

$$\mathbf{Re}\left\{\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)} + 1\right\} = \sum_{i=1}^{n} \alpha_i \mathbf{Re}\left(\frac{zf''_i(z)}{f'_i(z)} + 1\right). \tag{6}$$

Since $f_i \in K(\beta_i)$, we have

$$\mathbf{Re}\left(\frac{zf''_i(z)}{f'_i(z)} + 1\right) \geq \beta_i,$$

for all $i \in \{1, ..., n\}$.

We apply this property in (6) and we obtain that:

$$\mathbf{Re}\left\{\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)}+1\right\}=\sum_{i=1}^{n}\alpha_i\beta_i+1-\sum_{i=1}^{n}\alpha_i=\sum_{i=1}^{n}\alpha_i(\beta_i-1)+1. \quad (7)$$

The relation (7) implies that the integral operator $F_{\alpha_1,...,\alpha_n}$ is convex of order $\sum_{i=1}^{n}\alpha_i(\beta_i-1)+1$.

Theorem 5. *Let $\alpha_i \in \mathbf{R}, i \in \{1, ..., n\}$ and $\alpha_i > 0$. Consider $f_i \in UCV$, for all $i \in \{1, ..., n\}$. In these conditions, the integral operator $F_{\alpha_1,...,\alpha_n}$ is convex of order $1-\sum_{i=1}^{n}\alpha_i$, where $1-\sum_{i=1}^{n}\alpha_i \geq 0$.*

Proof. We have

$$\mathbf{Re}\left\{\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)}+1\right\}=\sum_{i=1}^{n}\alpha_i\mathbf{Re}\left(\frac{zf''_i(z)}{f'_i(z)}+1\right). \quad (8)$$

Because $f_i \in UCV$, we have

$$\mathbf{Re}\left\{1+\frac{zf''_i(z)}{f'_i(z)}\right\}\geq\left|\frac{zf''_i(z)}{f'_i(z)}\right|, z\in U.$$

We apply this inequality in relation (8) and obtain that:

$$\mathbf{Re}\left\{\frac{zF''_{\alpha_1,...,\alpha_n}(z)}{F'_{\alpha_1,...,\alpha_n}(z)}+1\right\}\geq\sum_{i=1}^{n}\alpha_i\left|\frac{zf''_i(z)}{f'_i(z)}\right|+1-\sum_{i=1}^{n}\alpha_i\geq 1-\sum_{i=1}^{n}\alpha_i. \quad (9)$$

The relation (9) implies that the integral operator $F_{\alpha_1,...,\alpha_n}$ is convex of order $1-\sum_{i=1}^{n}\alpha_i$.

References

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