# SOME PROPERTIES INVOLVING QUASI-CONVOLUTION PRODUCTS CONCERNING SOME SPECIAL CLASSES OF UNIVALENT FUNCTIONS

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Abstract. In this paper we give some sample properties involving quasi-convolution products of functions with negative coefficients.

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## 1. Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U$, $A = \{f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0\}$, $\mathcal{H}_u(U) = \{f \in \mathcal{H}(U) : f \text{ is univalent in } U\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

Let define the Alexander integral operator $I_A : A \to A$,

$$f(z) = I_A F(z) = \int_0^z \frac{F(t)}{t} dt \quad , z \in U \tag{1}$$

and the Bernardi integral operator $I_a : A \to A$,

$$f(z) = I_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt \quad , a = 1, 2, 3, ... \tag{2}$$

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It is easy to observe that for $f(z) \in A$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, we have $I_A f(z) = z + \sum_{j=2}^{\infty} \frac{a_j}{j} z^j$ and $I_a f(z) = z + \sum_{j=2}^{\infty} \frac{a+1}{a+j} a_j z^j$.

If we consider the functions $f(z), g(z) \in A$, $f(z) = z - \sum_{j=2}^{\infty} a_j \cdot z^j$ , $a_j \geq 0$, $j = 2, 3, ...$ $z \in U$ and $g(z) = z - \sum_{j=2}^{\infty} b_j \cdot z^j$ , $b_j \geq 0$ , $j = 2, 3, ...$ $z \in U$, we define the quasi-convolution product of the functions $f$ and $g$ by

$$(f * g)(z) = z - \sum_{j=2}^{\infty} a_j \cdot b_j \, z^j \, .$$

Similarly, the quasi-convolution product of more than two functions can also be defined. The quasi-convolution product was used in previously papers by Kumar (see [4], [5]), Owa (see [8], [9]), Misra (see [7]) and many others.

The purpose of this note is to obtain some properties regarding certain subclasses of functions with negative coefficients, by using the integral operators defined above and quasi-convolution products.

## 2. Preliminary results

**Definition 1** *[10] Let $T$ be the set of all functions $f \in S$ having the form:*

$$f(z) = z - \sum_{j=2}^{\infty} a_j \cdot z^j \ , \ a_j \geq 0 \ , \ j = 2, 3, ... \ \ z \in U \tag{3}$$

**Theorem 1** *[10] If $f \in A$, having the form (3), then the next two assertions are equivalent:*

$$(i) \qquad \sum_{j=2}^{\infty} j \cdot a_j \leq 1$$

$$(ii) \qquad f \in T$$

.

**Remark 1** *Using the definitions of the Alexander and Bernardi integral operators and the previously theorem, we observe that for* $f \in T$ *we have* $I_A(f) \in T$ *and* $I_a(f) \in T$. *More, we observe that for* $f \in T$, *having the form (3), we have* $a_j \in \left[0, \frac{1}{j}\right]$ *for all* $j \geq 2$. *If we denote* $T_{\left[0,\frac{1}{j}\right]} = \left\{ f(z) = z - \sum_{j=2}^{\infty} a_j \cdot z^j \ , \ a_j \in \left[0, \frac{1}{j}\right] \ , \ j = 2, 3, ... \ \ z \in U \right\}$, *we have* $T \subset T_{\left[0,\frac{1}{j}\right]}$. *Also,it is easy to observe that* $f \in T_{\left[0,\frac{1}{j}\right]}$ *we have* $I_A(f) \in T_{\left[0,\frac{1}{j}\right]}$ *and* $I_a(f) \in T_{\left[0,\frac{1}{j}\right]}$.

**Definition 2** *[2] A function* $f$ *is said to be uniformly starlike in the unit disc* $U$ *if* $f$ *is starlike and has the property that for every circular arc* $\gamma$ *contained in* $U$, *with center* $\alpha$ *also in* $U$, *the arc* $f(\gamma)$ *is starlike with respect to* $f(\alpha)$. *We let* $US^*$ *denote the class of all such functions. By taking* $UT^* = T \cap US^*$ *we define the class of uniformly starlike functions with negative coefficients.*

**Remark 2** *An arc* $f(\gamma)$ *is starlike with respect to a point* $w_0 = f(\alpha)$ *if* $arg(f(z) - w_0)$ *is nondecreasing as* $z$ *traces* $\gamma$ *in the positive direction.*

**Theorem 2** *[2] Let* $f \in S, f(z) = z + \sum_{n=2}^{\infty} a_n \cdot z^n, a_n \in C$. *If* $\sum_{n=2}^{\infty} n \cdot |a_n| \leq \frac{\sqrt{2}}{2}$ *then* $f \in US^*$.

**Definition 3** *[3] A function* $f$ *is uniformly convex in the disc* $U$ *if* $f \in S^C$ *(convex) and for every circular arc* $\gamma$ *contained in* $U$, *with center* $\alpha$ *also in* $U$ , *the arc* $f(\gamma)$ *is also convex. We let* $US^C$ *denote the class of all such functions. By taking* $UT^C = T \cap US^C$ *we define the class of uniformly convex functions with negative coefficients.*

**Remark 3** *The arc* $\gamma(t)$, $a < t < b$ *is convex if the argument of the tangent to* $\gamma(t)$ *is nondecreasing with respect to* $t$.

**Theorem 3** *[11] Let* $f \in T, f(z) = z - \sum_{n=2}^{\infty} a_n \cdot z^n$. *Then* $f \in UT^C$ *if and only if :*

$$\sum_{n=2}^{\infty} n(2n-1) \cdot a_n \leq 1.$$

**Remark 4** *In [6] the author showed that the Libera integral operator* $f : A \to A$ *defined by* $f(z) = I_1F(z) = \frac{2}{z}\int\limits_0^z F(t)dt$, $z \in U$ *preserve the class* $UT^*$. *Also, in [1] is showed that the Alexander integral operator, defined by (1), preserve the classes* $UT^*$, $UT^C$ *and the Bernardi integral operator, defined by (2), preserve the class* $UT^C$.

## 3. Main results

**Remark 5** *By using the series expansions of the functions* $f_1$, $f_2$, *the Theorem 1 and the definition of the quasi-convolution product it is easy to observe that for* $f_1 \in T$, $f_2 \in T_{\left[0,\frac{1}{j}\right]}$, *we have* $f_1 * f_2 \in T$.

**Theorem 4** *If* $f_1 \in T$, $f_2$, $f_3 \in T_{\left[0,\frac{1}{j}\right]}$, *then* $f_1 * f_2 * f_3 \in UT^C$.

*Proof.* Let $f_1(z) = z - \sum\limits_{j=2}^{\infty} a_j^1 \cdot z^j$, $a_j^1 \geq 0$, $j \geq 2$, $f_2(z) = z - \sum\limits_{j=2}^{\infty} a_j^2 \cdot z^j$, $a_j^2 \in \left[0, \frac{1}{j}\right]$, $j \geq 2$, $f_3(z) = z - \sum\limits_{j=2}^{\infty} a_j^3 \cdot z^j$, $a_j^3 \in \left[0, \frac{1}{j}\right]$, $j \geq 2$, and $(f_1 * f_2 * f_3)(z)$ $= z - \sum\limits_{j=2}^{\infty} c_j \cdot z^j$, where $c_j = a_j^1 \cdot a_j^2 \cdot a_j^3 \geq 0$, $j \geq 2$. From Remark 5 we have $f_1 * f_2 * f_3 \in T$.

From Theorem 3 we obtain that is sufficient to prove that

$$\sum_{j=2}^{\infty} j(2j-1) \cdot c_j \leq 1.$$

From $a_j^2, a_j^3 \in \left[0, \frac{1}{j}\right]$, $j \geq 2$ we have

$$j(2j-1) \cdot c_j = j(2j-1)a_j^1 \cdot a_j^2 \cdot a_j^3 \leq \frac{2j-1}{j^2} \cdot ja_j^1 < ja_j^1 . \tag{4}$$

Using Theorem 1 for the function $f_1$ we have

$$\sum_{j=2}^{\infty} ja_j^1 \leq 1 . \tag{5}$$

From (4) and (5) we obtain $\sum_{j=2}^{\infty} j(2j-1)\cdot c_j \leq 1$ . This mean that $f_1 * f_2 * f_3 \in UT^C$. Similarly, we obtain

**Corollary 1** *If* $f_1 \in T$ , $f_2$ , ... , $f_p \in T_{\left[0,\frac{1}{j}\right]}$, $p = 3, 4, \ldots$, *then* $f_1 * f_2 * \cdots * f_p \in UT^C$.

By using the Remark 4 and the Theorem 4 we obtain:

**Corollary 2** *If* $f_1 \in T$ , $f_2$ , $f_3 \in T_{\left[0,\frac{1}{j}\right]}$, *then* $I_A(f_1 * f_2 * f_3) \in UT^C$ *and* $I_a(f_1 * f_2 * f_3) \in UT^C$.

**Theorem 5** *If* $f_1 \in T$ , $f_2 \in T_{\left[0,\frac{1}{j}\right]}$, *then* $I_A(f_1 * f_2) \in UT^C$.

*Proof.* Let $f_1(z) = z - \sum_{j=2}^{\infty} a_j^1 \cdot z^j$ , $a_j^1 \geq 0$ , $j \geq 2$, $f_2(z) = z - \sum_{j=2}^{\infty} a_j^2 \cdot z^j$, $a_j^2 \in \left[0, \frac{1}{j}\right]$ , $j \geq 2$, and $I_A(f_1 * f_2)(z) = z - \sum_{j=2}^{\infty} c_j \cdot z^j$ , where $c_j = \frac{1}{j} \cdot a_j^1 \cdot a_j^2 \geq 0$ , $j \geq 2$. From Remark 5 we have $f_1 * f_2 \in T$, and from Remark 1 we obtain $I_A(f_1 * f_2) \in T$.

From Theorem 3 we obtain that is sufficient to prove that

$$\sum_{j=2}^{\infty} j(2j-1)\cdot c_j \leq 1.$$

From $a_j^2 \in \left[0, \frac{1}{j}\right]$ , $j \geq 2$ we have

$$j(2j-1)\cdot c_j = j(2j-1)\cdot \frac{1}{j}\cdot a_j^1 \cdot a_j^2 \leq \frac{2j-1}{j^2}\cdot ja_j^1 < ja_j^1 \,. \tag{6}$$

Using Theorem 1 for the function $f_1$ we have

$$\sum_{j=2}^{\infty} ja_j^1 \leq 1\,. \tag{7}$$

From (6) and (7) we obtain $\sum_{j=2}^{\infty} j(2j-1)\cdot c_j \leq 1$ . This mean that $I_A(f_1 * f_2) \in UT^C$. Similarly, we obtain

**Corollary 3** *If* $f_1 \in T$ , $f_2, \ldots, f_p \in T_{[0,\frac{1}{j}]}$, $p = 2, 3, \ldots,$ *then* $I_A(f_1 * f_2 * \cdots * f_p) \in UT^C$.

**Theorem 6** *If* $f_1 \in T$ , $f_2 \in T_{[0,\frac{1}{j}]}$, *then* $I_A(f_1) * I_A(f_2) \in UT^C$.

*Proof.* Let $f_1(z) = z - \sum_{j=2}^{\infty} a_j^1 \cdot z^j$ , $a_j^1 \geq 0$ , $j \geq 2$, $f_2(z) = z - \sum_{j=2}^{\infty} a_j^2 \cdot z^j$, $a_j^2 \in \left[0, \frac{1}{j}\right]$ , $j \geq 2$, and $I_A(f_1) * I_A(f_2)(z) = z - \sum_{j=2}^{\infty} c_j \cdot z^j$ , where $c_j = \frac{1}{j^2} \cdot a_j^1 \cdot a_j^2 \geq 0$ , $j \geq 2$. From Remarks 1 and 5 we obtain $I_A(f_1) * I_A(f_2) \in T$.

From Theorem 3 we obtain that is sufficient to prove that

$$\sum_{j=2}^{\infty} j(2j-1) \cdot c_j \leq 1.$$

From $a_j^2 \in [0,1)$ , $j \geq 2$ we have

$$j(2j-1) \cdot c_j = j(2j-1) \cdot \frac{1}{j^2} \cdot a_j^1 \cdot a_j^2 \leq \frac{2j-1}{j^2} \cdot j a_j^1 < j a_j^1 \,. \tag{8}$$

Using Theorem 1 for the function $f_1$ we have

$$\sum_{j=2}^{\infty} j a_j^1 \leq 1 \,. \tag{9}$$

From (8) and (9) we obtain $\sum_{j=2}^{\infty} j(2j-1) \cdot c_j \leq 1$ . This mean that $I_A(f_1) * I_A(f_2) \in UT^C$. Similarly, we obtain

**Corollary 4** *If* $f_1 \in T$ , $f_2, \ldots, f_p \in T_{[0,\frac{1}{j}]}$, $p = 2, 3, \ldots,$ *then* $I_A(f_1) * I_A(f_2) * \cdots * I_A(f_p) \in UT^C$.

**Theorem 7** *If* $f_1 \in T$ , $f_2 \in T_{[0,\frac{1}{j}]}$, *then* $f_1 * f_2 \in UT^*$.

*Proof.* Let $f_1(z) = z - \sum_{j=2}^{\infty} a_j \cdot z^j$ , $a_j \geq 0$ , $j \geq 2$, $f_2(z) = z - \sum_{j=2}^{\infty} b_j \cdot z^j$, $b_j \in \left[0, \frac{1}{j}\right]$ , $j \geq 2$, and $(f_1 * f_2)(z) = z - \sum_{j=2}^{\infty} c_j \cdot z^j$ , $c_j = a_j \cdot b_j \geq 0$ , $j \geq 2$. From Remark 5 we have $f_1 * f_2 \in T$. In this conditions, from Theorem 2, we obtain that is sufficient to prove that $\sum_{j=2}^{\infty} \sqrt{2} \cdot j \cdot c_j \leq 1$.

Using $f_1 \in T$, from Theorem 1 we have

$$\sum_{j=2}^{\infty} j \cdot a_j \leq 1. \tag{10}$$

Using $b_j \in \left[0, \frac{1}{j}\right]$ , $j \geq 2$, we have

$$\sqrt{2} \cdot j \cdot c_j = \sqrt{2} \cdot j \cdot a_j \cdot b_j \leq j \cdot a_j \cdot \frac{\sqrt{2}}{j} < j \cdot a_j \, . \tag{11}$$

From (10) and (11) we obtain $\sum_{j=2}^{\infty} \sqrt{2} \cdot j \cdot c_j \leq 1$ . This mean that $f_1 * f_2 \in UT^*$.Similarly, we obtain

**Corollary 5** *If* $f_1 \in T$ , $f_2$ , ... , $f_p \in T_{\left[0, \frac{1}{j}\right]}$, $p = 2, 3, \ldots$, *then* $f_1 * f_2 * \cdots * f_p \in UT^*$.

By using Remark 1 and Theorem 7 we obtain

**Theorem 8** *If* $f_1 \in T$ , $f_2 \in T_{\left[0, \frac{1}{j}\right]}$, *then* $I_A(f_1) * I_A(f_2) \in UT^*$ *and* $I_a(f_1) * I_a(f_2) \in UT^*$.

**Corollary 6** *If* $f_1 \in T$ , $f_2$ , ... , $f_p \in T_{\left[0, \frac{1}{j}\right]}$, $p = 2, 3, \ldots$, *then* $I_A(f_1) * I_A(f_2) * \cdots * I_A(f_p) \in UT^*$ *and* $I_a(f_1) * I_a(f_2) * \cdots * I_a(f_p) \in UT^*$.

By using Remark 4 and Theorem 7 we obtain

**Theorem 9** *If* $f_1 \in T$ , $f_2 \in T_{\left[0, \frac{1}{j}\right]}$, *then* $I_A(f_1 * f_2) \in UT^*$.

**Corollary 7** *If* $f_1 \in T$ , $f_2$ , ... , $f_p \in T_{[0,\frac{1}{j}]}$, $p = 2, 3, \dots$, *then* $I_A(f_1 * f_2 * \dots * f_p) \in UT^*$.

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