Acta Universitatis Apulensis
ISSN: 1582-5329
http://www.uab.ro/auajournal/

No. 78/2024
pp. 121-146
doi: 0.17114/j.aua.2024.78.09

A NOTE ON CONFORMAL BI-SLANT SUBMERSIONS WITH VERTICAL REEB VECTOR FIELD-ξ

M. Shuaib and T. Fatima

Abstract. This paper explores the potential for conformal bi-slant submersions from a Sasakian manifold, specifically addressing the condition where the Reeb vector field-ξ is vertical. Given that bi-slant submersions inherently involve slant distributions, we investigate the integrability of these distributions and the geometry of the corresponding distribution leaves. Additionally, the concept of pluriharmonicity is examined, with illustrative evidence provided to support the findings of this paper.

2010 *Mathematics Subject Classification*: 53D10, 53C43.

Keywords: "Sasakian manifold", "Riemannian submersions" "Conformal bi-slant submersions".

1. Introduction

B. O'Neill [25] and A. Gray [14] were the ones who first proposed and developed the concept of submersions and immersions. For Riemannian manifolds, they discovered certain Riemannian equations by studying the geometrical characteristics. Submersions theory becomes fascinating topic in differential geometry to discuss the properties between differentiable structures. Riemannian submersions is the subject of study throughout both mathematics and physics since it has numerous applications, most notably in Kaluza-Klein theory and Yang-Mills theory (see [9], [41], [23], [20]). B. Watson [40] investigated the Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds in the year 1976. Later, B Sahin [32] studied geometric characteristics and Riemannian submersions geometry. Using an almost Hermitian manifold, he defined anti-invariant Riemannian submersions onto Riemannian manifolds. He demonstrates that under the almost complex structure of the total manifold, their vertical distribution is anti-invariant. Many authors looked into and expanded on this research by studying anti-invariant submersions [4], [32], semi-invariant submersions [33], slant submersions [12], [34], and semi-slant submersions [18], [26], among other topics. Tastan, Sahin, and Yanan [38] defined and investigated hemi-slant submersions from almost Hermitian manifolds as a generalisation case of semi-invariant and semi-slant submersions.

From almost Hermitian to almost contact metric manifolds, D. Chinea [10] expanded the notion of Riemannian submersions. He examined base space, total, and fibre space from an instrinsic geometric perspective point. R. Prasad extended the concept of hemi-slant submersions a step further, by defining quasi-bi-slant submersions from almost contact metric manifold [28], [27]. The results he obtained for submersions were interesting, and he also discovered some decomposition theorems.

B. Fuglede [15] and T. Ishihara [21] introduced the concept of conformal submersion as a generalisation of Riemannian submersions and talked about some of their geometric characteristics. It is clear that conformal submersion with dilation $\lambda = 1$ is a Riemannian submersion. Gudmundsson and Wood [17] investigated conformal holomorphic submersion as a generalisation of holomorphic submersion. They were able to obtain the necessary and sufficient conditions for harmonic morphisms of conformal holomorphic submersions. Later on, conformal anti-invariant submersions , [35], [29], conformal semi-invariant submersions [5], conformal slant submersions [3], and conformal semi-slant submersions [2] have been studied and defined by Akyol and Sahin. A number of researchers have recently explored the geometry of conformal hemi-slant submersions [36], [37], conformal bi-slant submersions [6] , and quasi bi-slant conformal submersions [7] and they have discussed some decomposition theorems. Additionally, they expanded the idea of pluriharmonicity from almost Hermitian manifolds to almost contact metric manifolds.

In this paper, we investigate conformal bi-slant submersions from Sasakian manifold onto a Riemannian manifold with vertical vector field ξ. The structure of the paper is as follows. Section 2 introduces almost contact metric manifolds, precisely the Sasakian manifold with the properties required for this study. Our paper's third section includes a definition of conformal bi-slant submersion as well as some noteworthy discoveries. Section 4 detailed the conditions needed for distribution integrability as well as the total geodesicness of its leaves. This section also discusses how a total space becomes a locally twisted product manifold. Finally, at the end of the study, the concept of ϕ-pluriharmonicity is addressed.

Note: Throughout the paper, we will consider abbreviations as follows:
Almost contact metric manifold-$\mathcal{ACM}$ manifold
Conformal bi-slant submersion- $\mathcal{CBSS}$
Riemannian manifold - $\mathcal{RM}$
Riemannian submersion - $\mathcal{RS}$
Sasakian manifold- $\mathcal{SM}$

2. Preliminaries

We start off by providing a few definitions and findings that will be quite helpful for our research and will aid in exploring the central idea of the research paper.

Definition 1. *[8] Let $\mathcal{J}$ be a $\mathcal{RS}$ from an $\mathcal{ACM}$ manifold $(\bar{O}_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ $(\bar{O}_2, g_2)$. Then $\mathcal{J}$ is called a horizontally conformal submersion, if there is a positive function λ such that*

$$g_1(U_1, V_1) = \frac{1}{\lambda^2} g_2(\mathcal{J}_* U_1, \mathcal{J}_* V_1), \tag{1}$$

for any $U_1, V_1 \in \Gamma(ker \mathcal{J}_*)^\perp$. It is obvious that every $\mathcal{RS}$ is a particularly horizontally conformal submersion with $\lambda = 1$.

Let $\mathcal{J} : \bar{O}_1 \to \bar{O}_2$ be a $\mathcal{RS}$. For any horizontal vector field $\bar{W}$, if $\mathcal{J}$-related with $\bar{W}$ on $\bar{O}_2$ i.e $\mathcal{J}_*(\bar{W}(q)) = \bar{W}\mathcal{J}(q)$, for $q \in \Gamma(T\bar{O}_1)$, then $\bar{W}$ is said to basic vector field.

The formulae for tensor fields $\mathcal{T}$ and $\mathcal{A}$ given by :

$$\mathcal{T}(L_1, L_2) = \mathcal{T}_{L_1} L_2 = \mathcal{H}\nabla_{\mathcal{V}L_1}\mathcal{V}L_2 + \mathcal{V}\nabla_{\mathcal{V}L_1}\mathcal{H}L_2 \tag{2}$$

$$\mathcal{A}(L_1, L_2) = \mathcal{A}_{L_1} L_2 = \mathcal{V}\nabla_{\mathcal{H}L_1}\mathcal{H}L_2 + \mathcal{H}\nabla_{\mathcal{H}L_1}\mathcal{V}L_2 \tag{3}$$

for all vector fields $L_1, L_2 \in \Gamma(T\bar{O}_1)$ [13].

On using (2) and (3), we get

$$\nabla_{\bar{W}_1}\bar{Z}_1 = \mathcal{T}_{\bar{W}_1}\bar{Z}_1 + \bar{\nabla}_{\bar{W}_1}\bar{Z}_1 \tag{4}$$

$$\nabla_{\bar{W}_1}\bar{X}_1 = \mathcal{T}_{\bar{W}_1}\bar{X}_1 + \mathcal{H}\nabla_{\bar{W}_1}\bar{X}_1 \tag{5}$$

$$\nabla_{\bar{X}_1}\bar{W}_1 = \mathcal{A}_{\bar{X}_1}\bar{W}_1 + \mathcal{V}\nabla_{\bar{X}_1}\bar{W}_1 \tag{6}$$

$$\nabla_{\bar{X}_1}\bar{Y}_1 = \mathcal{H}\nabla_{\bar{X}_1}\bar{Y}_1 + \mathcal{A}_{\bar{X}_1}\bar{Y}_1 \tag{7}$$

for all $\bar{W}_1, \bar{Z}_1 \in \Gamma(\ker \mathcal{J}_*)$ and $\bar{X}_1, \bar{Y}_1 \in \Gamma(\ker \mathcal{J}_*)^\perp$ where $\bar{\nabla}_{\bar{W}_1}\bar{Z}_1 = \mathcal{V}\nabla_{\bar{W}_1}\bar{Z}_1$. Then $\mathcal{T}_{\bar{Z}}$ and $\mathcal{A}_{\bar{W}}$ are skew-symmetric, i.e.,

$$g(\mathcal{A}_{\bar{W}}F_1, F_2) = -g(F_1, \mathcal{A}_{\bar{W}}F_2), \quad g(\mathcal{T}_{\bar{Z}}F_1, F_2) = -g(F_1, \mathcal{T}_{\bar{Z}}F_2) \tag{8}$$

for all $F_1, F_2 \in \Gamma(T_p\bar{O}_1)$.

Now, we recall the definition of weakly horizontal conformal submersions.

Proposition 1. *[16] Let $\mathcal{J} : \bar{O}_1 \to \bar{O}_2$ be horizontally conformal submersion with dilation λ and $\bar{Z}, \bar{W} \in \Gamma(\ker \mathcal{J}_*)^\perp$, then*

$$\mathcal{A}_{\bar{W}}\bar{Z} = \frac{1}{2}(\mathcal{V}[\bar{W}, \bar{Z}] - \lambda^2 g_1(\bar{W}, \bar{Z}) \operatorname{grad}_v \frac{1}{\lambda^2}) \tag{9}$$

The second fundamental form of $\mathcal{J}$ is defined by

$$(\nabla \mathcal{J}_*)(\bar{W}, \bar{Z}) = \nabla^{\mathcal{J}}_{\bar{W}} \mathcal{J}_* \bar{Z} - \mathcal{J}_* \nabla_{\bar{W}} \bar{Z} \tag{10}$$

A map is said to be totally geodesic if $(\nabla \mathcal{J}_*)(\bar{W}, \bar{Z}) = 0$ for all $\bar{W}, \bar{Z} \in \Gamma(T_p \bar{O}_1)$, where Levi-Civita and pullback connections are ∇ and $\nabla^{\mathcal{J}}$. [39].

Lemma 1. *Let $\mathcal{J} : \bar{O}_1 \to \bar{O}_2$ be a horizontal conformal submersion. Then, we have*

(i) $(\nabla \mathcal{J}_*)(\bar{W}, \bar{Z}) = \bar{W}(\ln \lambda)\mathcal{J}_*(\bar{Z}) + \bar{Z}(\ln \lambda)\mathcal{J}_*(\bar{W}) - g_1(\bar{W}, \bar{Z})\mathcal{J}_*(grad \ln \lambda)$,

(ii) $(\nabla \mathcal{J}_*)(\bar{U}, \bar{V}) = -\mathcal{J}_*(\mathcal{T}_{\bar{U}} \bar{V})$,

(iii) $(\nabla \mathcal{J}_*)(\bar{W}, \bar{U}) = -\mathcal{J}_*(\nabla^{\mathcal{J}}_{\bar{W}} \bar{U}) = -\mathcal{J}_*(\mathcal{A}_{\bar{W}} \bar{U})$.

for any horizontal vector fields $\bar{W}, \bar{Z}$ and vertical vector fields $\bar{U}, \bar{V}$ [8].

Here, we giving the definition of twisted product manifold given by R. Ponge [30]. Let $(\bar{O}_1, g)$ be a $\mathcal{RM}$ with M_1 and M_2 be submanifolds of $\bar{O}_1$. Then a product manifold of the form $\bar{O}_1 = M_1 \times_\lambda M_2$ is said to be a twisted product manifold if and only if M_1 is a totally geodesic foliation and M_2 is a totally umbilic foliation

Let M be a differentiable manifold of dimension n, is said to be having an almost contact structure (ϕ, ξ, η) if, it carries a tensor field ϕ, vector field ξ and 1-form η on M satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1, \tag{11}$$

where, I is identity tensor. The almost contact structure (ϕ, ξ, η) is said to be normal if $N + d\eta \otimes \xi = 0$, where N is the Nijenhuis tensor of ϕ. Suppose that a Riemannian metric tensor g is given in M and satisfies the condition

$$g(\phi\bar{W}, \phi\bar{Z}) = g(\bar{W}, \bar{Z}) - \eta(\bar{W})\eta(\bar{Z}), \quad \eta(\bar{W}) = g(\bar{W}, \xi), \tag{12}$$

for all $\bar{Z}, \bar{W} \in \Gamma(TM)$. Then (ϕ, ξ, η, g)-structure is called an $\mathcal{ACM}$ structure. Let Φ be the fundamental 2-form on M, i.e, $\Phi(\bar{U}, \bar{V}) = g(\bar{U}, \phi\bar{V})$. If $\Phi = d\eta$, M is said to be a contact manifold. A normal contact metric structure is called a Sasakian structure, which satisfies

$$(\nabla_{\bar{W}} \phi)\bar{Z} = g(\bar{W}, \bar{Z})\xi - \eta(Z)W \tag{13}$$

where ∇ is the Levi-Civita connection of g. From above formula, we have for $\mathcal{SM}$

$$\nabla_{\bar{W}} \xi = -\phi\bar{W}, \tag{14}$$

and the covariant derivative of ϕ is defined by

$$(\nabla_{\bar{W}} \phi)\bar{Z} = \nabla_{\bar{W}} \phi\bar{Z} - \phi\nabla_{\bar{W}} \bar{Z}, \tag{15}$$

for any vector fields $\bar{W}, \bar{Z} \in \Gamma(TM)$.

3. Conformal bi-slant submersions ($\mathcal{CBSS}$)

Definition 2. *Let* $(\bar{O}_1, \phi, \xi, \eta, g_1)$ *be an* $\mathcal{ACM}$ *manifold and* $(\bar{O}_2, g_2)$ *be a* $\mathcal{RM}$. *A conformal submersion* $\mathcal{J}$ *is said to be a* $\mathcal{CBSS}$ *if* D_{θ_1} *and* D_{θ_2} *are slant distributions with slant angel* θ_1 *and* θ_2 *such that* $\ker \mathcal{J}_* = D_{\theta_1} \oplus D_{\theta_2} \oplus < \xi >$, *where* $< \xi >$ *is a* 1*-dimensional distribution spanned by* ξ *and* $\mathcal{J}$ *is called proper if* $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

If $n_1,\ n_2 \neq 0,\ 0 < \theta_1 < \frac{\pi}{2}$ and $0 < \theta_2 < \frac{\pi}{2}$ then, $\mathcal{J}$ is said to be proper $\mathcal{CBSS}$, where n_1, n_2 are the dimensions of D_{θ_1} and D_{θ_2} respectively.

In this part, we provide a non-trivial example to support our research. Note that $\mathbb{R}^{2n+1}$ denote a $\mathcal{SM}$ with the structure (ϕ, ξ, η, g) defined as

$$\phi\left(\sum_{i=1}^{n}\left(U_i\frac{\partial}{\partial u^i} + V_i\frac{\partial}{\partial v^i}\right) + W\frac{\partial}{\partial w}\right) = \sum_{i=1}^{n}\left(V_i\frac{\partial}{\partial u^i} - U_i\frac{\partial}{\partial v^i}\right),$$

$$\eta = \frac{1}{2}\left(dw - \sum_{i=1}^{n} v^i du^i\right), \xi = 2\frac{\partial}{\partial w}$$

$$g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}\left(du^i \otimes du^i + dv^i \otimes dv^i\right),$$

where $\left(u^1, \ldots, u^n, v^1, \ldots, v^n, w\right)$ are the Cartesian coordinates.

Now, taking into account the definition above, we can provide the following examples:.

Example 1. *Let* $\mathcal{J} : (\mathbb{R}^9, g_{\mathbb{R}^9}) \longrightarrow (\mathbb{R}^5, g_{\mathbb{R}^5})$ *be a conformal submersion defined by*

$$\begin{aligned}&\mathcal{J}\left(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, w\right)\\ &= \pi^6\left(\cos\theta_1 u_1 - \sin\theta_1 u_3, \frac{u_2 + u_4}{\sqrt{2}}, \sin\theta_2 v_3 + \cos\theta_2 v_4, v_1, w\right)\end{aligned}$$

then

$$\begin{aligned}\ker F_* =&< W_1 = \sin\theta_1\partial u_1 + \cos\theta_1\partial u_3, W_2 = \frac{1}{\sqrt{2}}\left(\partial u_2 - \partial u_4\right)\\ &W_3 = \cos\theta_2\partial v_3 - \sin\theta_2\partial v_4, W_4 = \partial v_2, W_5 = \xi = \partial w > \quad \textit{and}\\ (\ker F_*)^{\perp} =&< Z_1 = \cos\theta_1\partial u_1 - \sin\theta_1\partial u_3, Z_2 = \frac{1}{\sqrt{2}}\left(\partial u_2 + \partial u_4\right)\\ &Z_3 = \sin\theta_2\partial v_3 + \cos\theta_2\partial v_4, Z_4 = \partial v_1 >\end{aligned}$$

Thus, the submersion $\mathcal{J}$ *is* $\mathcal{CBSS}$ *with* $D_{\theta_1} = \langle W_1, W_3\rangle$ *with slant angle* $\bar{\theta}_1$ *such that* $\cos\bar{\theta}_1 = \sin(\theta_2 - \theta_1)$ *and* $D_{\theta_2} = \langle W_2, W_4\rangle$ *with the slant angle* $\bar{\theta}_2 = \dfrac{\pi}{4}$.

Example 2. *Let* $\mathcal{J} : (\mathbb{R}^9, g_{\mathbb{R}^9}) \longrightarrow (\mathbb{R}^5, g_{\mathbb{R}^5})$ *such that*

$$(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4, w) = e^5 \left(\frac{u_1 + \sqrt{3}u_2}{2}, \sin\theta_1 u_3 + \cos\theta_1 u_4, v_1, v_2, w \right).$$

Then it follows that

$$\begin{aligned} D_{\theta_1} &= < W_1 = \frac{1}{2}\left(\sqrt{3}\partial u_1 - \partial u_2\right), W_3 = \partial v_3, W_5 = \xi = \partial w >, \\ D_{\theta_2} &= \operatorname{span}\langle W_2 = \cos\theta_1 \partial u_3 - \sin\theta_1 \partial u_4, W_4 = \partial v_4 \rangle \quad \textit{and} \\ (\ker F_*)^{\perp} &= < X_1 = \frac{1}{2}(\partial u_1 + \sqrt{3}\partial u_2), X_2 = \sin\theta_1 \partial u_3 + \cos\theta_1 \partial u_4, \\ & \quad X_3 = \partial v_1, X_4 = \partial v_2 > . \end{aligned}$$

Hence, $\mathcal{J}$ *is a* $\mathcal{CBSS}$ *with the slant angles* $\bar{\theta}_1 = \frac{\pi}{3}$ *and* $\bar{\theta}_2 = \theta_1$*, respectively and* $\lambda = e^{\sqrt{5}}$.

Suppose that $\mathcal{J}$ is a $\mathcal{CBSS}$ from $\mathcal{SM}$ $(\bar{O}_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ $(\bar{O}_2, g_2)$, then for any $\bar{U} \in \ker \mathcal{J}_*$

$$\bar{U} = \mathfrak{K}\bar{U} + \mathfrak{L}\bar{U} + \eta(\bar{U})\xi \tag{16}$$

where $\mathfrak{K}\bar{U} \in \Gamma(D_{\theta_1})$ and $\mathfrak{L}\bar{U} \in \Gamma(D_{\theta_2})$.

Also, for $\bar{U} \in \Gamma(\ker \mathcal{J}_*)$

$$\phi\bar{U} = \omega\bar{U} + \zeta\bar{U} \tag{17}$$

where $\omega\bar{U} \in \Gamma(\ker \mathcal{J}_*)$ and $\zeta\bar{U} \in \Gamma(\ker \mathcal{J}_*)^{\perp}$. For any $\bar{X} \in \Gamma(\ker \mathcal{J}_*)^{\perp}$, we have

$$\phi\bar{X} = t\bar{X} + f\bar{X} \tag{18}$$

where $t\bar{X} \in \Gamma(\ker \mathcal{J}_*)$ and $f\bar{X} \in \Gamma(\ker \mathcal{J}_*)^{\perp}$.

The horizontal distribution $(\ker \mathcal{J}_*)^{\perp}$ is decomposed as

$$(\ker \mathcal{J}_*)^{\perp} = \zeta D_{\theta_1} \oplus \zeta D_{\theta_2} \oplus \mu \tag{19}$$

such that μ is the complementary distribution to $\zeta D_{\theta_1} \oplus \zeta D_{\theta_2}$ in $\Gamma(\ker \mathcal{J}_*)^{\perp}$.

Given that $\mathcal{J} : \bar{O}_1 \to \bar{O}_2$ is a $\mathcal{CBSS}$, let's present some insightful findings that will be used throughout the work.

Theorem 2. *Let* $\mathcal{J} : (\bar{O}_1, \phi, \xi, \eta, g_1) \to (\bar{O}_2, g_2)$ *be a* $\mathcal{CBSS}$ *from* $\mathcal{ACM}$ *manifold onto a* $\mathcal{RM}$ *with slant angles* θ_1 *and* θ_2*. Then we have*

(i) $\omega^2\bar{W} = -(\cos^2\theta_1)\bar{W}$

(ii) $g_1(\omega\bar{W}, \omega\bar{Z}) = cos^2\theta_1 g_1(\bar{W}, \bar{Z})$

(iii) $g_1(\zeta\bar{W}, \zeta\bar{Z}) = sin^2\theta_1 g_1(\bar{W}, \bar{Z})$,

for any vector fields $\bar{W}, \bar{Z} \in \Gamma(D_{\theta_1})$

Proof. Let $\mathcal{J}$ be a $\mathcal{CBSS}$ from $\mathcal{ACM}$ manifold $(\bar{O_1}, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ $(\bar{O_2}, g_2)$. Then for any $\bar{W} \in \Gamma(D_{\theta_1})$, we have

$$\cos\theta_1 = \frac{|\omega\bar{W}|}{|\phi\bar{W}|} \tag{20}$$

$$\cos\theta_1 = \frac{g_1(\phi\bar{W}, \omega\bar{W})}{|\phi\bar{W}||\omega\bar{W}|}.$$

From equation (38), we have

$$\cos\theta_1 = -\frac{g_1(\bar{W}, \omega^2\bar{W})}{|\phi\bar{W}||\omega\bar{W}|}. \tag{21}$$

By using (11), (20) and (21), we get

$$\omega^2 U = (-\cos^2\theta_1)\bar{W}$$

. For (ii) part, for any vector field $\bar{W}, \bar{Z} \in \Gamma(D_{\theta_1})$ with using equation (17), we have

$$g_1(\omega\bar{W}, \omega\bar{Z}) = g_M(\phi\bar{W}, \omega\bar{Z}).$$

Taking account fact from (17) and from Theorem 2, we get

$$g_1(\omega\bar{W}, \omega\bar{Z}) = \cos^2\theta_1 g_1(\bar{W}, \bar{Z}). \tag{22}$$

Again, by using equations (17), we have

$$g_1(\zeta U, \zeta V) = g_1(\phi\bar{W}, \phi\bar{Z}) - g_1(\phi\bar{W}, \omega\bar{Z}).$$

From equations (17) (22)and , we have

$$g_1(\zeta\bar{W}, \zeta\bar{Z}) = \sin^2\theta_1 g_1(\bar{W}, \bar{Z}),$$

from which we get result.

In a similar way, we can provide the following result.

Theorem 3. *Let* $\mathcal{J} : (\bar{O_1}, \phi, \xi, \eta, g_1) \to (\bar{O_2}, g_2)$ *be a* $\mathcal{CBSS}$ *from* $\mathcal{ACM}$ *manifold onto a* $\mathcal{RM}$ *with slant angles* θ_1 *and* θ_2*. Then we have* *(i)* $\omega^2\bar{U} = -(\cos^2\theta_2)\bar{U}$

(ii) $g_1(\omega\bar{U}, \omega\bar{V}) = cos^2\theta_2 g_1(\bar{U}, \bar{V})$

(iii) $g_1(\zeta\bar{U}, \zeta\bar{V}) = sin^2\theta_1 g_1(\bar{U}, \bar{V})$,

for any vector fields $\bar{U}, \bar{V} \in \Gamma(D_{\theta_2})$

Proof. The proof of this is similar to the proof of Theorem 2. Therefore, we skip the proof.

Lemma 4. *Let* $(\bar{O}_1, \phi, \xi, \eta, g_1)$ *be a* $\mathcal{ACM}$ *manifold and* $(\bar{O}_2, g_2)$ *be a* $\mathcal{RM}$*. If* $\mathcal{J} : \bar{O}_1 \to \bar{O}_2$ *is a* $\mathcal{CBSS}$*, then we have*

$$\zeta t\bar{X} + f^2\bar{X} = -\bar{X}, \quad \omega t\bar{X} + tf\bar{X} = 0$$

$$-\bar{U} + \eta(\bar{U})\xi = \omega^2\bar{U} + t\zeta\bar{U}, \quad \zeta\omega\bar{U} + f\zeta\bar{U} = 0$$

for any $\bar{U} \in \Gamma(\ker\mathcal{J}_*)$ *and* $\bar{X} \in \Gamma(\ ker\mathcal{J}_*)^{\perp}$.

Proof. Equations (14), (17) and (18) are used to obtain outcomes from simple calculations.

Let $(\bar{O}_2, g_2)$ is a $\mathcal{RM}$ and that $(\bar{O}_1, \phi, \xi, \eta, g_1)$ is a $\mathcal{SM}$. We now consider how the Sasakian structure on $\bar{O}_1$ affects the tensor fields $\mathcal{T}$ and $\mathcal{A}$ of a $\mathcal{BSCS}$ $\mathcal{J} : (\bar{O}_1, \phi, \xi, \eta, g_1) \to (\bar{O}_2, g_2)$

Lemma 5. *the If* $\mathcal{J} : \bar{O}_1 \to \bar{O}_2$ *is a* $\mathcal{CBSS}$ *where* $(\bar{O}_1, \phi, \xi, \eta, g_1)$ *be a* $\mathcal{SM}$ *and* $(\bar{O}_2, g_2)$ *be a* $\mathcal{RM}$*. , then we have*

$$\mathcal{A}_{\bar{X}}t\bar{Y} + \mathcal{H}\nabla_{\bar{X}}f\bar{Y} = f\mathcal{H}\nabla_{\bar{X}}\bar{Y} + \zeta\mathcal{A}_{\bar{X}}\bar{Y} \tag{23}$$

$$\mathcal{V}\nabla_{\bar{X}}t\bar{Y} + \mathcal{A}_{\bar{X}}f\bar{Y} = t\mathcal{H}\nabla_{\bar{X}}\bar{Y} + \omega\mathcal{A}_{\bar{X}}\bar{Y} - g_1(\bar{X}, \bar{Y})\xi \tag{24}$$

$$\mathcal{V}\nabla_{\bar{X}}\omega\bar{V} + \mathcal{A}_{\bar{X}}\zeta\bar{V} = t\mathcal{A}_{\bar{X}}\bar{V} + \omega\mathcal{V}\nabla_{\bar{X}}\bar{V} + g_1(\bar{X}, \bar{V})\xi \tag{25}$$

$$\mathcal{A}_{\bar{X}}\omega\bar{V} + \mathcal{H}\nabla_{\bar{X}}\zeta\bar{V} = f\mathcal{A}_{\bar{X}}\bar{V} + \zeta\mathcal{V}\nabla_{\bar{X}}\bar{V} - \eta(\bar{V})\bar{X} \tag{26}$$

$$\mathcal{V}\nabla_{\bar{V}}t\bar{X} + \mathcal{T}_{\bar{V}}f\bar{X} = \omega\mathcal{T}_{\bar{V}}f\bar{X} + t\mathcal{H}\nabla_{\bar{V}}\bar{X} + g_1(\bar{V}, \bar{X})\xi \tag{27}$$

$$\mathcal{T}_{\bar{V}}t\bar{X} + \mathcal{H}\nabla_{\bar{V}}f\bar{X} = \zeta\mathcal{T}_{\bar{V}}\bar{X} + f\mathcal{H}\nabla_{\bar{V}}\bar{X} \tag{28}$$

$$\mathcal{V}\nabla_{\bar{U}}\omega\bar{V} + \mathcal{T}_{\bar{U}}\zeta\bar{V} + \eta(\bar{V})\bar{U} = t\mathcal{T}_{\bar{U}}\bar{V} + \omega\mathcal{V}\nabla_{\bar{U}}\bar{V} + g_1(\bar{U}, \bar{V})\xi \tag{29}$$

$$\mathcal{T}_{\bar{U}}\omega\bar{V} + \mathcal{H}\nabla_{\bar{U}}\zeta\bar{V} = f\mathcal{T}_{\bar{U}}\bar{V} + \zeta\mathcal{V}\nabla_{\bar{U}}\bar{V} \tag{30}$$

for any $\bar{U}, \bar{V} \in \Gamma(\ker\mathcal{J}_*)$ *and* $\bar{X}, \bar{Y} \in \Gamma(\ker\mathcal{J}_*)^{\perp}$.

Proof. By some simple steps of calculation with using equations (18), (7) and (15), (23) and (24) easily obtained. In the same manner, from equations (17), (18), (4)-(6) and (15), we will get the desired results.

We will now go through some fundamental findings that can be used to investigate the $\mathcal{CBSS}$ $\mathcal{J} : \bar{O}_1 \to \bar{O}_2$ geometry. Define the following in this regard:

$$(\nabla_{\bar{U}}\omega)\bar{V} = \mathcal{V}\nabla_{\bar{U}}\omega\bar{V} - \omega\mathcal{V}\nabla_{\bar{U}}\bar{V} \tag{31}$$

$$(\nabla_{\bar{U}}\zeta)\bar{V} = \mathcal{H}\nabla_{\bar{U}}\zeta\bar{V} - \zeta\mathcal{V}\nabla_{\bar{U}}\bar{V} \tag{32}$$

$$(\nabla_{\bar{X}}t)\bar{Y} = \mathcal{V}\nabla_{\bar{X}}t\bar{Y} - t\mathcal{H}\nabla_{\bar{X}}\bar{Y} \tag{33}$$

$$(\nabla_{\bar{X}}f)\bar{Y} = \mathcal{H}\nabla_{\bar{X}}f\bar{Y} - f\mathcal{H}\nabla_{\bar{X}}\bar{Y} \tag{34}$$

for all $\bar{U}, \bar{V} \in \Gamma(\ker \mathcal{J}_*)$ and $\bar{X}, \bar{Y} \in \Gamma(\ker \mathcal{J}_*)^{\perp}$.

Lemma 6. *Let $(\bar{O}_1, \phi, \xi, \eta, g_1)$ be a $\mathcal{SM}$ and $(\bar{O}_2, g_2)$ be a $\mathcal{RM}$. If $\mathcal{J} : \bar{O}_1 \to \bar{O}_2$ is a $\mathcal{CBSS}$, then we have*

$$(\nabla_{\bar{U}}\omega)\bar{V} = t\mathcal{T}_{\bar{U}}\bar{V} - \mathcal{T}_{\bar{U}}\zeta\bar{V} + g_1(\bar{U}, \bar{V})\xi - \eta(\bar{V})\bar{U}$$

$$(\nabla_{\bar{U}}\zeta)\bar{V} = f\mathcal{T}_{\bar{U}}\bar{V} - \mathcal{T}_{\bar{U}}\omega\bar{V}$$

$$(\nabla_{\bar{X}}t)\bar{Y} = \omega\mathcal{A}_{\bar{X}}\bar{Y} - \mathcal{A}_{\bar{X}}f\bar{Y} + g_1(X, Y)\xi$$

$$(\nabla_{\bar{X}}f)\bar{Y} = \zeta\mathcal{A}_{\bar{X}}\bar{Y} - \mathcal{A}_{\bar{X}}t\bar{Y},$$

for any $\bar{U}, \bar{V} \in \Gamma(\ker\mathcal{J}_*)$ and $\bar{X}, \bar{Y} \in \Gamma(\ker\mathcal{J}_*)^{\perp}$.

Proof. By taking account the fact of equations (13), (4)-(7), (23), (24), (29), (30), equations (31)-(34), it is easy to get the proof of the lemma.

It is Given that the ∇ is the connection of $\mathcal{SM}$ $\bar{O}_1$. let us suppose that the tenors ω and ζ are parallel, we can write

$$t\mathcal{T}_{\bar{U}}\bar{V} = \mathcal{T}_{\bar{U}}\zeta\bar{V} - g_1(\bar{U}, \bar{V})\xi + \eta(\bar{V})\bar{U}, \quad f\mathcal{T}_{\bar{U}}\bar{V} = \mathcal{T}_{\bar{U}}\omega\bar{V},$$

for any $\bar{X}, \bar{Y} \in \Gamma(T\bar{O}_1)$.

4. Geometry of leaves of distributions

We studied the concept of $\mathcal{CBSS}$ from $\mathcal{SM}$ in this present paper. As per the idea that to ensure the presence of two slant distributions D_{θ_1} and D_{θ_1}, it is extremely significant to investigate the integrability specifications for slant distributions.

Theorem 7. *Let us suppose that $\mathcal{J}:(\bar{O}_1,\phi,\xi,\eta,g_1)\to(\bar{O}_2,g_2)$ be a $\mathcal{CBSS}$, where $(\bar{O}_1,\phi,\xi,\eta,g_1)$ is a Sasakian manifold. Then D_{θ_1} is not integrable.*

Proof. By using the equation $g_1([U_1,V_1],\xi)$ for $U_1,V_1\in\Gamma(D_{\theta_1})$ with taking account from equation (14), we get $g_1(\nabla_{U_1}V_1-\nabla_{V_1}U_1,\xi)=2g_1(\phi U_1,V_1)\neq 0$. It clear that $g_1([U_1,V_1],\xi)\neq 0$. Hence, the slant distribution D_{θ_1} is not integrable.

We assume that the Reeb vector field ξ is vertical throughout the study. The above conclusion demonstrates that distribution D_{θ_1} is not integrable. We can solve this problem if we figure out the integrability specifications for distribution D_{θ_2}.

Theorem 8. *Let $\mathcal{J}:(\bar{O}_1,\phi,\xi,\eta,g_1)\to(\bar{O}_2,g_{\bar{O}_2})$ be a proper $\mathcal{CBSS}$ with slant angles θ_1 and θ_2, where $(\bar{O}_1,\phi,\xi,\eta,g_1)$ is a Kenmotsu manifold and $(\bar{O}_2,g_2)$ is a $\mathcal{RM}$. Then the distribution $D_{\theta_1}\oplus<\xi>$ is integrable if and only if*

$$\begin{aligned}&\lambda^{-2}g_2((\nabla\mathcal{J}_*)(\bar{U}_1,\zeta\bar{V}_1),\mathcal{J}_*\zeta\bar{U}_2)\\&=\lambda^{-2}\{(g_2(\nabla^{\mathcal{J}}_{\bar{U}_1}\mathcal{J}_*\zeta\bar{V}_1-\nabla^{\mathcal{J}}_{\bar{V}_1}\mathcal{J}_*\zeta\bar{U}_1),\mathcal{J}_*\zeta\bar{U}_2)\}+g_1(T_{\bar{V}_1}\zeta\omega\bar{U}_1-T_{\bar{U}_1}\zeta\omega\bar{V}_1,\bar{U}_2)\\&\quad+g_1(T_{\bar{U}_1}\zeta\bar{V}_1-T_{\bar{V}_1}\zeta\bar{U}_1,\omega\bar{U}_2)+\lambda^{-2}g_2((\nabla\mathcal{J}_*)(\bar{V}_1,\zeta\bar{U}_1),\mathcal{J}_*\zeta\bar{U}_2)\\&\quad+g_1(\bar{V}_1-\bar{U}_1,\phi\bar{U}_2+\xi),\end{aligned}$$

for any vector fields $\bar{U}_1,\bar{V}_1\in\Gamma(D_{\theta_1}\oplus<\xi>)$ and $\bar{U}_2\in\Gamma(D_{\theta_2})$

Proof. For any $\bar{U}_1,\bar{V}_1\in\Gamma(D_{\theta_1}\oplus<\xi>)$ and $\bar{U}_2\in\Gamma(D_{\theta_2})$ and on using equations (12), (13) and from (17), we have

$$\begin{aligned}g_1([\bar{U}_1,\bar{V}_1],\bar{U}_2)=&\,g_1(\nabla_{\bar{V}_1}\omega^2\bar{U}_1,\bar{U}_2)-g_1(\nabla_{\bar{U}_1}\omega^2\bar{V}_1,\bar{U}_2)-g_1(\nabla_{\bar{U}_1}\zeta\omega\bar{V}_1,\bar{U}_2)\\&+g_1(\nabla_{\bar{V}_1}\zeta\omega\bar{U}_1,\bar{U}_2)+g_1(\nabla_{\bar{U}_1}\zeta\bar{V}_1,\phi\bar{U}_2)-g_1(\nabla_{\bar{V}_1}\zeta\bar{U}_1,\phi\bar{U}_2)\\&+g_1(\bar{V}_1-\bar{U}_1,\xi)g_1(\bar{V}_1-\bar{U}_1,\phi\bar{U}_2).\end{aligned}$$

Considering Theorem 2, we have

$$\begin{aligned}\sin^2\theta_1 g_1([\bar{U}_1,\bar{V}_1],\bar{U}_2)=&-g_1(\nabla_{\bar{U}_1}\zeta\omega\bar{V}_1,\bar{U}_2)+g_1(\nabla_{\bar{V}_1}\zeta\omega\bar{U}_1,\bar{U}_2)+g_1(\nabla_{\bar{U}_1}\zeta\bar{V}_1,\phi\bar{U}_2)\\&-g_1(\nabla_{\bar{V}_1}\zeta\bar{U}_1,\phi\bar{U}_2)+g_1(\bar{V}_1-\bar{U}_1,\phi\bar{U}_2+\xi).\end{aligned}$$

On using equation (5), we have

$$\begin{aligned}\sin^2\theta_1 g_1([\bar{U}_1,\bar{V}_1],\bar{U}_2)=&\,g_1(\mathcal{T}_{\bar{V}_1}\zeta\omega\bar{U}_1-\mathcal{T}_{\bar{U}_1}\zeta\omega\bar{V}_1,\bar{U}_2)-g_1(\mathcal{T}_{\bar{U}_1}\zeta\bar{V}_1-\mathcal{T}_{\bar{V}_1}\zeta\bar{U}_1,\omega\bar{U}_2)\\&+g_1(\mathcal{H}\nabla_{\bar{U}_1}\zeta\bar{V}_1-\mathcal{H}\nabla_{\bar{V}_1}\zeta\bar{U}_1,\zeta\bar{U}_2)+g_1(\bar{V}_1-\bar{U}_1,\phi\bar{U}_2+\xi).\end{aligned}$$

Now considering Lemma 1 and equation (10), we yields

$$\begin{aligned}
&\sin^2\theta_1 g_1([\bar{U}_1,\bar{V}_1],\bar{U}_2)\\
&= \lambda^{-2} g_2((\nabla^{\mathcal{J}}_{\bar{U}_1}\mathcal{J}_*\zeta\bar{V}_1 - \nabla^{\mathcal{J}}_{\bar{V}_1}\mathcal{J}_*\zeta\bar{U}_1),\mathcal{J}_*\zeta\bar{U}_2)\} + g_1(\mathcal{T}_{\bar{V}_1}\zeta\omega\bar{U}_1 - \mathcal{T}_{\bar{U}_1}\zeta\omega\bar{V}_1,\bar{U}_2)\\
&\quad - g_1(\mathcal{T}_{\bar{U}_1}\zeta\bar{V}_1 - \mathcal{T}_{\bar{V}_1}\zeta\bar{U}_1,\omega\bar{U}_2) - \lambda^{-2} g_2((\nabla\mathcal{J}_*)(\bar{U}_1,\zeta\bar{V}_1),\mathcal{J}_*\zeta\bar{U}_2)\\
&\quad + \lambda^{-2} g_2((\nabla\mathcal{J}_*)(\bar{V}_1,\zeta\bar{U}_1),\mathcal{J}_*\zeta\bar{U}_2) + g_1(\bar{V}_1 - \bar{U}_1,\phi\bar{U}_2 + \xi).
\end{aligned}$$

In same manner, we study the necessary and sufficient conditions under which slant distribution D_{θ_2} is integrable.

Theorem 9. *Let us suppose that $\mathcal{J} : (\bar{O}_1,\phi,\xi,\eta,g_1) \to (\bar{O}_2,g_2)$ be a $\mathcal{CBSS}$, where $(\bar{O}_1,\phi,\xi,\eta,g_1)$ is a $\mathcal{SM}$. Then D_{θ_2} is not integrable.*

Proof. By using the equation $g_1([U_2,V_2],\xi)$ for $U_2,V_2 \in \Gamma(D_{\theta_2})$ with taking account from equation (14), we get $g_1(\nabla_{U_2}V_2 - \nabla_{V_2}U_2,\xi) = 2g_1(\phi U_2,V_2) \neq 0$. It clear that $g_1([U_2,V_2],\xi) \neq 0$. Hence, the slant distribution D_{θ_2} is not integrable.

Above mentioned result shows that the same condition with slant distribution D_{θ_2} for integrability, so again we discuss the necessary and sufficient condition of integrability for distribution $D_{\theta_2}\oplus <\xi>$ as follows :

Theorem 10. *Let $\mathcal{J} : (M,\phi,\xi,\eta,g_1) \to (\bar{O}_2,g_2)$ be a proper $\mathcal{CBSS}$ with slant angles θ_1 and θ_2, where (M,ϕ,ξ,η,g_1) is a Kenmotsu manifold and $(\bar{O}_2,g_2)$ is a $\mathcal{RM}$. Then the distribution $D_{\theta_2}\oplus <\xi>$ is integrable if and only if*

$$\begin{aligned}
&\lambda^{-2} g_2((\nabla\mathcal{J}_*)(\bar{W},\zeta\bar{V}_2),\mathcal{J}_*\zeta\bar{U}_2)\\
&= \lambda^{-2}\{(g_2(\nabla^{\mathcal{J}}_{\bar{W}}\mathcal{J}_*\zeta\bar{V}_2 - \nabla^{\mathcal{J}}_{\bar{V}_2}\mathcal{J}_*\zeta\bar{U}_2),\mathcal{J}_*\zeta\bar{U}_1)\} + g_1(T_{\bar{V}_2}\zeta\omega\bar{W} - T_{\bar{W}}\zeta\omega\bar{V}_2,\bar{W})\\
&\quad + g_1(T_{\bar{W}}\zeta\bar{V}_2 - T_{\bar{V}_2}\zeta\bar{W},\omega\bar{W}) + \lambda^{-2} g_2((\nabla\mathcal{J}_*)(\bar{V}_2,\zeta\bar{W}),\mathcal{J}_*\zeta\bar{W})\\
&\quad + g_1(\bar{V}_2 - \bar{U}_2,\phi\bar{U}_1 + \xi)
\end{aligned}$$

for any $\bar{U}_2 \in \Gamma(D_{\theta_1})$ and $\bar{U}_2,\bar{V}_2 \in \Gamma(D_{\theta_2}\oplus <\xi>)$.

Proof. For any $\bar{U}_2,\bar{V}_2 \in \Gamma(D_{\theta_1}\oplus <\xi>)$ and $\bar{W} \in \Gamma(D_{\theta_2})$ and on using equations (12), (13) and from (17), we have

$$\begin{aligned}
g_1([\bar{U}_2,\bar{V}_2],\bar{W}) &= g_1(\nabla_{\bar{V}_2}\omega^2\bar{U}_2,\bar{W}) - g_1(\nabla_{\bar{U}_2}\omega^2\bar{V}_2,\bar{W}) - g_1(\nabla_{\bar{U}_2}\zeta\omega\bar{V}_2,\bar{W})\\
&\quad + g_1(\nabla_{\bar{V}_2}\zeta\omega\bar{U}_2,\bar{W}) + g_1(\nabla_{\bar{U}_2}\zeta\bar{V}_2,\phi\bar{W}) - g_1(\nabla_{\bar{V}_2}\zeta\bar{U}_2,\phi\bar{W})\\
&\quad + g_1(\bar{V}_1 - \bar{U}_1,\xi)g_1(\bar{V}_1 - \bar{U}_1,\phi\bar{U}_2).
\end{aligned}$$

Considering Theorem 2, we have

$$\sin^2\theta_1 g_1([\bar{U}_2,\bar{V}_2],\bar{W}) = -g_1(\nabla_{\bar{U}_2}\zeta\omega\bar{V}_2,\bar{W}) + g_1(\nabla_{\bar{V}_2}\zeta\omega\bar{U}_2,\bar{W}) + g_1(\nabla_{\bar{U}_2}\zeta\bar{V}_2,\phi\bar{W}) - g_1(\nabla_{\bar{V}_2}\zeta\bar{U}_2,\phi\bar{W}) + g_1(\bar{V}_1-\bar{U}_1,\phi\bar{U}_2+\xi).$$

On using equation (5), we have

$$\begin{aligned}&\sin^2\theta_1 g_1([\bar{U}_2,\bar{V}_2],\bar{W})\\ &= g_1(\mathcal{T}_{\bar{V}_2}\zeta\omega\bar{U}_2 - \mathcal{T}_{\bar{U}_2}\zeta\omega\bar{V}_2,\bar{W}) - g_1(\mathcal{T}_{\bar{U}_2}\zeta\bar{V}_2 - \mathcal{T}_{\bar{V}_2}\zeta\bar{U}_2,\omega\bar{W})\\ &\quad + g_1(\mathcal{H}\nabla_{\bar{U}_2}\zeta\bar{V}_2 - \mathcal{H}\nabla_{\bar{V}_2}\zeta\bar{U}_2,\zeta\bar{W}) + g_1(\bar{V}_1-\bar{U}_1,\phi\bar{U}_2+\xi).\end{aligned}$$

Now considering Lemma 1 and equation (10), we yields

$$\begin{aligned}&\sin^2\theta_1 g_1([\bar{U}_2,\bar{V}_2],\bar{W})\\ &= \lambda^{-2}g_2((\nabla^{\mathcal{J}}_{\bar{U}_2}\mathcal{J}_*\zeta\bar{V}_1 - \nabla^{\mathcal{J}}_{\bar{V}_2}\mathcal{J}_*\zeta\bar{U}_1),\mathcal{J}_*\zeta\bar{U}_2)\} + g_1(\mathcal{T}_{\bar{V}_2}\zeta\omega\bar{U}_2 - \mathcal{T}_{\bar{U}_2}\zeta\omega\bar{V}_2,\bar{W})\\ &\quad - g_1(\mathcal{T}_{\bar{U}_2}\zeta\bar{V}_2 - \mathcal{T}_{\bar{V}_2}\zeta\bar{U}_2,\omega\bar{W}) - \lambda^{-2}g_2((\nabla\mathcal{J}_*)(\bar{U}_2,\zeta\bar{V}_2),\mathcal{J}_*\zeta\bar{W})\\ &\quad + \lambda^{-2}g_2((\nabla\mathcal{J}_*)(\bar{V}_2,\zeta\bar{U}_2),\mathcal{J}_*\zeta\bar{W}) + g_1(\bar{V}_1-\bar{U}_1,\phi\bar{U}_2+\xi).\end{aligned}$$

Studying distribution leaves will be significant since they are crucial to the geometry of $\mathcal{CBSS}$ from the $\mathcal{SM}$. In order to do this, we are determining the circumstances in which distributions define totally geodesic foliation on M.

Theorem 11. *Let $\mathcal{J}:(\bar{O}_1,\phi,\xi,\eta,g_1)\to(\bar{O}_2,g_2)$ be a $\mathcal{CBSS}$ from a $\mathcal{SM}$ onto a $\mathcal{RM}$ $\bar{O}_2$. Then D_{θ_1} is not totally geodesic on M.*

Proof. By taking the vector field $\bar{U},\bar{V}\in\Gamma(D_{\theta_1})$ and since $\bar{V}$ and ξ are orthogonal, we have

$$g_1(\nabla_{\bar{U}}\bar{V},\xi) = -g_1(\bar{V},\nabla_{\bar{U}}\xi)$$

By considering the equations (14), we get

$$g_1(\nabla_{\bar{U}}\bar{V},\xi) = g_1(\phi\bar{U},\bar{V}).$$

Since, $\bar{U},\bar{V}\in\Gamma(D_{\theta_1}), g_1(\phi\bar{U},\bar{V})\neq 0$, that is $g_1(\nabla_{\bar{U}}\bar{V},\xi)\neq 0$. Hence, the distribution is not defines totally geodesic foliation on $\bar{O}_1$.

Since the Reeb vector field ξ is assumed to be vertical, the slant distribution D_{θ_1} does not defines totally geodesic foliation. In order to deal with this issue, the geometry of the leaves of the slant distribution $D_{\theta_1}\oplus <\xi>$ is being examined here.

Theorem 12. *Let $(\bar{O}_1, \phi, \xi, \eta, g_1)$ be a $\mathcal{SM}$ and $(\bar{O}_2, g_2)$ be a $\mathcal{RM}$ such that $\mathcal{J}$ is a $\mathcal{CBSS}$ from M onto $\bar{O}_2$. Then the distribution $D_{\theta_1} \oplus < \xi >$ is defines totally geodesic foliation on M if and only if*

$$\begin{aligned}&\lambda^{-2} g_2((\nabla \mathcal{J}_*)(\bar{U}_1, \zeta \bar{V}_1), \mathcal{J}_* \zeta \bar{U}_2) + \eta(\bar{V}_1) g_1(\phi \bar{U}_2, \bar{U}_1)\\ &= g_1(\mathcal{T}_{\bar{U}_1} \zeta \bar{V}_1, \omega \bar{U}_2) - g_1(\mathcal{T}_{\bar{U}_1} \zeta \omega \bar{V}_1, \bar{U}_2) + \lambda^{-2} g_2(\nabla^{\mathcal{J}}_{\bar{U}_1} \mathcal{J}_* \zeta \bar{V}_1, \mathcal{J}_* \zeta \bar{U}_2),\end{aligned} \tag{35}$$

and

$$\begin{aligned}&\lambda^{-2} g_2(\nabla^{\mathcal{J}}_{\bar{X}} \mathcal{J}_* \zeta \bar{U}_1, \mathcal{J}_* \zeta \bar{V}_1) + g_1(\mathcal{A}_{\bar{X}} \zeta \bar{U}_1, \omega \bar{V}_1) - g_1(\bar{U}_1 + \bar{V}_1, \xi) g_1(t\bar{X}, \bar{U}_1 + \bar{V}_1)\\ &= \sin^2 \theta g_1([\bar{U}_1, \bar{X}], \bar{V}_1) + g_1(\mathcal{A}_{\bar{X}} \zeta \omega \bar{U}_1, \bar{V}_1) + g_1(grad \ln \lambda, \bar{X}) g_1(\zeta \bar{U}_1, \zeta \bar{V}_1)\\ &\quad + g_1(grad \ln \lambda, \zeta \bar{U}_1) g_1(\bar{X}, \zeta \bar{V}_1) - g_1(grad \ln \lambda, \zeta \bar{V}_1) g_1(\bar{X}, \zeta \bar{U}_1),\end{aligned} \tag{36}$$

for any $\bar{U}_1, \bar{V}_1 \in \Gamma(D_{\theta_1} \oplus < \xi >), \bar{U}_2 \in \Gamma(D_{\theta_2})$ and $\bar{X} \in \Gamma(\ker \mathcal{J}_*)^{\perp}$.

Proof. For any $\bar{U}_1, \bar{V}_1 \in \Gamma(D_{\theta_1} \oplus < \xi >)$ and $\bar{U}_2 \in \Gamma(D_{\theta_2})$ with using equations (12), (13) and (17), we have

$$\begin{aligned}g_1(\nabla_{\bar{U}_1} \bar{V}_1, \bar{U}_2) &= g_1(\nabla_{\bar{U}_1} \zeta \bar{V}_1, \phi \bar{U}_2) - g_1(\nabla_{\bar{U}_1} \zeta \omega \bar{V}_1, \bar{U}_2)\\ &\quad - g_1(\nabla_{\bar{U}_1} \omega^2 \bar{V}_1, \bar{U}_2) - \eta(\bar{V}_1) g_1(\phi \bar{U}_2, \bar{U}_1).\end{aligned}$$

From Theorem 2, we can write

$$\sin^2 \theta_1 g_1(\nabla_{\bar{U}_1} \bar{V}_1, \bar{U}_2) = -g_1(\nabla_{\bar{U}_1} \zeta \omega \bar{V}_1, \bar{U}_2) + g_1(\nabla_{\bar{U}_1} \zeta \bar{V}_1, \phi \bar{U}_2) - \eta(\bar{V}_1) g_1(\phi \bar{U}_2, \bar{U}_1).$$

On using equation (5), we have

$$\begin{aligned}\sin^2 \theta_1 g_1(\nabla_{\bar{U}_1} \bar{V}_1, \bar{U}_2) &= g_1(\mathcal{T}_{\bar{U}_1} \zeta \bar{V}_1, \omega \bar{U}_2) - g_1(\mathcal{T}_{\bar{U}_1} \zeta \omega \bar{V}_1, \bar{U}_2)\\ &\quad + g_1(\mathcal{H} \nabla_{\bar{U}_1} \zeta \bar{V}_1, \zeta \bar{U}_2) - \eta(\bar{V}_1) g_1(\phi \bar{U}_2, \bar{U}_1).\end{aligned}$$

Considering equation (10) and Lemma 1, we obtain

$$\begin{aligned}&\sin^2 \theta_1 g_1(\nabla_{\bar{U}_1} \bar{V}_1, \bar{U}_2)\\ &= g_1(\mathcal{T}_{\bar{U}_1} \zeta \bar{V}_1, \omega \bar{U}_2) - g_1(\mathcal{T}_{\bar{U}_1} \zeta \omega \bar{V}_1, \bar{U}_2) - \lambda^{-2} g_2((\nabla \mathcal{J}_*)(\bar{U}_1, \zeta \bar{V}_1), \mathcal{J}_* \zeta \bar{U}_2)\\ &\quad + \lambda^{-2} g_2(\nabla_{\bar{U}_1} \mathcal{J}_* \zeta \bar{V}_1, \mathcal{J}_* \zeta \bar{U}_2) - \eta(\bar{V}_1) g_1(\phi \bar{U}_2, \bar{U}_1),\end{aligned}$$

which is the equation first in Theorem 12.

On the other hand, On using (12), (13) and (17), we can write

$$\begin{aligned}g_1(\nabla_{\bar{U}_1} \bar{V}_1, \bar{X}) &= -g_1([\bar{U}_1, \bar{X}], \bar{V}_1) + g_1(\phi \nabla_{\bar{X}} \omega \bar{U}_1, \bar{V}_1) - g_1(\nabla_{\bar{X}} \zeta \bar{U}_1, \phi \bar{V}_1)\\ &\quad + g_1(\bar{U}_1 + \bar{V}_1, \xi) g_1(t\bar{X}, \bar{U}_1 + \bar{V}_1),\end{aligned}$$

for any $\bar{U}_1, \bar{V}_1 \in \Gamma(D_{\theta_1})$ and $\bar{X} \in \Gamma(\ker \mathcal{J}_*)^{\perp}$ .Considering Theorem 2, we obtained

$$\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1, \bar{X}) = -g_1([\bar{U}_1, \bar{V}_1], \bar{X}) + g_1(\nabla_{\bar{X}}\zeta\omega\bar{U}_1, \bar{V}_1) \\ - g_1(\nabla_{\bar{X}}\zeta\bar{U}_1, \phi\bar{V}_1) + g_1(\bar{U}_1 + \bar{V}_1, \xi)g_1(t\bar{X}, \bar{U}_1 + \bar{V}_1).$$

On using equation (7), we have

$$\begin{aligned} &\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1, \bar{X}) \\ &= \sin^2\theta_1 g_1([\bar{U}_1, \bar{X}], \bar{V}_1) + g_1(\mathcal{A}_{\bar{X}}\zeta\omega\bar{U}_1, \bar{V}_1) - g_1(\mathcal{A}_{\bar{X}}\zeta\bar{U}_1, \omega\bar{V}_1) \\ &\quad - \lambda^{-2}g_2(\mathcal{J}_*\nabla_{\bar{X}}\zeta\bar{U}_1, \mathcal{J}_*\zeta\bar{U}_1) + g_1(\bar{U}_1 + \bar{V}_1, \xi)g_1(t\bar{X}, \bar{U}_1 + \bar{V}_1) \end{aligned}$$

Using Lemma 1, we yields

$$\begin{aligned} &\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1, \bar{X}) \\ &= \sin^2\theta_1 g_1([\bar{U}_1, \bar{X}], \bar{V}_1) + g_1(\mathcal{A}_{\bar{X}}\zeta\omega\bar{U}_1, \bar{V}_1) - \lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{X}}\mathcal{J}_*\zeta\bar{U}_1, \mathcal{J}_*\zeta\bar{V}_1) \\ &\quad + g_1(grad\ln\lambda, \bar{X})g_1(\zeta\bar{U}_1, \zeta\bar{V}_1) + g_1(grad\ln\lambda, \zeta\bar{U}_1)g_1(\bar{X}, \zeta\bar{V}_1) \\ &\quad - g_1(grad\ln\lambda, \zeta\bar{V}_1)g_1(\bar{X}, \zeta\bar{U}_1) - g_1(\mathcal{A}_{\bar{X}}\zeta\bar{U}_1, \omega\bar{V}_1) \\ &\quad + g_1(\bar{U}_1 + \bar{V}_1, \xi)g_1(t\bar{X}, \bar{U}_1 + \bar{V}_1). \end{aligned}$$

This completes the proof of the Theorem.

Theorem 13. *Let $\mathcal{J} : (\bar{O}_1, \phi, \xi, \eta, g_1) \to (\bar{O}_2, g_2)$ be a $\mathcal{CBSS}$ from a $\mathcal{SM}$ onto a $\mathcal{RM}$ $\bar{O}_2$. Then D_{θ_2} is not defines totally geodesic on $\bar{O}_1$.*

Proof. By taking the vector field $\bar{X}, \bar{Y} \in \Gamma(D_{\theta_2})$ and since $\bar{Y}$ and ξ are orthogonal, we have

$$g_1(\nabla_{\bar{X}}\bar{Y}, \xi) = -g_1(\bar{Y}, \nabla_{\bar{X}}\xi)$$

By considering the equations (14), we get

$$g_1(\nabla_{\bar{X}}\bar{Y}, \xi) = g_1(\phi\bar{X}, \bar{Y}).$$

Since, $\bar{X}, \bar{Y} \in \Gamma(D_{\theta_2}), g_1(\phi\bar{X}, \bar{Y}) \neq 0$, that is $g_1(\nabla_{\bar{X}}\bar{Y}, \xi) \neq 0$. Hence, the distribution is not defines totally geodesic foliation on $\bar{O}_1$.

Since, again distribution D_{θ_2} not defines totally geodesic foliations. In same manner, we examine the conditions for distribution $D_{\theta_2} \oplus < \xi >$ is totally geodesic.

Theorem 14. *Let $(M, \phi, \xi, \eta, g_1)$ be a $\mathcal{SM}$ and $(\bar{O}_2, g_2)$ be a $\mathcal{RM}$ such that $\mathcal{J}$ is a $\mathcal{CBSS}$ from M onto $\bar{O}_2$. Then the distribution $D_{\theta_2} \oplus < \xi >$ is defines totally geodesic foliation on M if and only if*

$$\begin{aligned} &\lambda^{-2}g_2((\nabla\mathcal{J}_*)(\bar{U}_2, \zeta\bar{V}_2), \mathcal{J}_*\zeta\bar{W}) + \eta(\bar{V}_2)g_1(\phi\bar{U}_2, \bar{U}_2) \\ &= g_1(\mathcal{T}_{\bar{U}_2}\zeta\bar{V}_2, \omega\bar{W}) - g_1(\mathcal{T}_{\bar{U}_2}\zeta\omega\bar{V}_2, \bar{W}) + \lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{U}_2}\mathcal{J}_*\zeta\bar{V}_1, \mathcal{J}_*\zeta\bar{U}_2), \end{aligned} \tag{37}$$

and

$$\begin{aligned}&\lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{Y}}\mathcal{J}_*\zeta\bar{U}_2,\mathcal{J}_*\zeta\bar{V}_2)+g_1(\mathcal{A}_{\bar{Y}}\zeta\bar{U}_2,\omega\bar{V}_2)-g_1(\bar{U}_2+\bar{V}_2,\xi)g_1(t\bar{X},\bar{U}_2+\bar{V}_2)\\&=\sin^2\theta g_1([\bar{U}_2,\bar{Y}],\bar{V}_2)+g_1(\mathcal{A}_{\bar{Y}}\zeta\omega\bar{U}_2,\bar{V}_2)+g_1(grad\ln\lambda,\bar{Y})g_1(\zeta\bar{U}_2,\zeta\bar{V}_2)\\&\quad+g_1(grad\ln\lambda,\zeta\bar{U}_2)g_1(\bar{Y},\zeta\bar{V}_2)-g_1(grad\ln\lambda,\zeta\bar{V}_2)g_1(\bar{Y},\zeta\bar{U}_2),\end{aligned}\tag{38}$$

for any $\bar{U}_2,\bar{V}_2\in\Gamma(D_{\theta_2}\oplus<\xi>),\bar{W}\in\Gamma(D_{\theta_1})$ *and* $\bar{Y}\in\Gamma(\ker\mathcal{J}_*)^{\perp}$.

Proof. For any $\bar{U}_2,\bar{V}_2\in\Gamma(D_{\theta_2}\oplus<\xi>)$ and $\bar{W}\in\Gamma(D_{\theta_1})$ with using equations (12), (13) and (17), we have

$$\begin{aligned}g_1(\nabla_{\bar{U}_2}\bar{V}_2,\bar{W})&=g_1(\nabla_{\bar{U}_2}\zeta\bar{V}_2,\phi\bar{W})-g_1(\nabla_{\bar{U}_2}\zeta\omega\bar{V}_2,\bar{W})\\&\quad-g_1(\nabla_{\bar{U}_2}\omega^2\bar{V}_2,\bar{W})-\eta(\bar{V}_2)g_1(\phi\bar{U}_2,\bar{U}_2).\end{aligned}$$

From Theorem 2, we can write

$$\sin^2\theta_1g_1(\nabla_{\bar{U}_2}\bar{V}_2,\bar{W})=-g_1(\nabla_{\bar{U}_2}\zeta\omega\bar{V}_2,\bar{W})+g_1(\nabla_{\bar{U}_2}\zeta\bar{V}_2,\phi\bar{W})-\eta(\bar{V}_2)g_1(\phi\bar{U}_2,\bar{U}_2).$$

On using equation (5), we have

$$\begin{aligned}\sin^2\theta_1g_1(\nabla_{\bar{U}_2}\bar{V}_2,\bar{W})&=g_1(\mathcal{T}_{\bar{U}_2}\zeta\bar{V}_2,\omega\bar{W})-g_1(\mathcal{T}_{\bar{U}_2}\zeta\omega\bar{V}_2,\bar{W})\\&\quad+g_1(\mathcal{H}\nabla_{\bar{U}_2}\zeta\bar{V}_2,\zeta\bar{W})-\eta(\bar{V}_2)g_1(\phi\bar{U}_2,\bar{U}_2).\end{aligned}$$

Considering equation (10) and Lemma 1, we obtain

$$\begin{aligned}&\sin^2\theta_1g_1(\nabla_{\bar{U}_2}\bar{V}_2,\bar{W})\\&=g_1(\mathcal{T}_{\bar{U}_2}\zeta\bar{V}_2,\omega\bar{W})-g_1(\mathcal{T}_{\bar{U}_2}\zeta\omega\bar{V}_2,\bar{W})-\lambda^{-2}g_2((\nabla\mathcal{J}_*)(\bar{U}_2,\zeta\bar{V}_2),\mathcal{J}_*\zeta\bar{W})\\&\quad+\lambda^{-2}g_2(\nabla_{\bar{U}_2}\mathcal{J}_*\zeta\bar{V}_1,\mathcal{J}_*\zeta\bar{U}_2)-\eta(\bar{V}_2)g_1(\phi\bar{U}_2,\bar{U}_2),\end{aligned}$$

which is the equation first in Theorem 14.

On the other hand, $\bar{U}_2,\bar{V}_2\in\Gamma(D_{\theta_1})$ and $\bar{Y}\in\Gamma(\ker\mathcal{J}_*)^{\perp}$ with using (12), (13) and (17), we can write

$$\begin{aligned}g_1(\nabla_{\bar{U}_2}\bar{V}_2,\bar{Y})&=-g_1([\bar{U}_2,\bar{Y}],\bar{V}_2)+g_1(\phi\nabla_{\bar{Y}}\omega\bar{U}_2,\bar{V}_2)-g_1(\nabla_{\bar{Y}}\zeta\bar{U}_2,\phi\bar{V}_2)\\&\quad+g_1(\bar{U}_2+\bar{V}_2,\xi)g_1(t\bar{X},\bar{U}_2+\bar{V}_2).\end{aligned}$$

Considering Theorem 2, we obtained

$$\begin{aligned}\sin^2\theta_1g_1(\nabla_{\bar{U}_2}\bar{V}_2,\bar{Y})&=-g_1([\bar{U}_2,\bar{V}_2],\bar{Y})+g_1(\nabla_{\bar{Y}}\zeta\omega\bar{U}_2,\bar{V}_2)\\&\quad-g_1(\nabla_{\bar{Y}}\zeta\bar{U}_2,\phi\bar{V}_2)+g_1(\bar{U}_2+\bar{V}_2,\xi)g_1(t\bar{X},\bar{U}_2+\bar{V}_2).\end{aligned}$$

On using equation (7), we have

$$\begin{aligned}\sin^2\theta_1 g_1(\nabla_{\bar{U}_2}\bar{V}_2,\bar{Y}) = \sin^2\theta_1 g_1([\bar{U}_2,\bar{Y}],\bar{V}_2) + g_1(\mathcal{A}_{\bar{Y}}\zeta\omega\bar{U}_2,\bar{V}_2) - g_1(\mathcal{A}_{\bar{Y}}\zeta\bar{U}_2,\omega\bar{V}_2)\\ - \lambda^{-2}g_2(\mathcal{J}_*\nabla_{\bar{Y}}\zeta\bar{U}_2,\mathcal{J}_*\zeta\bar{U}_2) + g_1(\bar{U}_2+\bar{V}_2,\xi)g_1(t\bar{X},\bar{U}_2+\bar{V}_2)\end{aligned}$$

Using Lemma 1, we yields

$$\begin{aligned}&\sin^2\theta_1 g_1(\nabla_{\bar{U}_2}\bar{V}_2,\bar{Y})\\ &= \sin^2\theta_1 g_1([\bar{U}_2,\bar{Y}],\bar{V}_2) + g_1(\mathcal{A}_{\bar{Y}}\zeta\omega\bar{U}_2,\bar{V}_2) - \lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{Y}}\mathcal{J}_*\zeta\bar{U}_2,\mathcal{J}_*\zeta\bar{V}_2)\\ &+ g_1(grad\ln\lambda,\bar{Y})g_1(\zeta\bar{U}_2,\zeta\bar{V}_2) + g_1(grad\ln\lambda,\zeta\bar{U}_2)g_1(\bar{Y},\zeta\bar{V}_2)\\ &- g_1(grad\ln\lambda,\zeta\bar{V}_2)g_1(\bar{Y},\zeta\bar{U}_2) - g_1(\mathcal{A}_{\bar{Y}}\zeta\bar{U}_2,\omega\bar{V}_2) + g_1(\bar{U}_2+\bar{V}_2,\xi)g_1(t\bar{X},\bar{U}_2+\bar{V}_2).\end{aligned}$$

This completes the proof of the Theorem.

now, we study the geometry of leaves of vertical distributions. We commence by giving the findings for vertical distribution.

Theorem 15. *Let $\mathcal{J} : (\bar{O}_1,\phi,\xi,\eta,g_1) \to (\bar{O}_2,g_2)$ be $\mathcal{CBSS}$ from a $\mathcal{SM}$ $(\bar{O}_1,\phi,\xi,\eta,g_1)$ onto a $\mathcal{RM}$ $(\bar{O}_2,g_2)$. Then the vertical distribution $(ker\mathcal{J}_*)$ is not defines totally geodesic foliation on $\bar{O}_1$.*

Proof. Taking $\bar{W} \in \Gamma(ker\mathcal{J}_*)^{\perp}$ and $\bar{V} \in \Gamma(ker\mathcal{J}_*)$, $g_1(\nabla_{\bar{V}}\xi,\bar{W})$. Taking account the fact of equation (14), we have, $g_1(\phi\bar{V},\bar{W}) \neq 0$, that is $g_1(\nabla_{\bar{V}}\xi,\bar{W}) \neq 0$. Hence, vertical distribution $ker\mathcal{J}_*$ is not totally geodesic foliation on $\bar{O}_1$.

Now, that we have a one-dimensional distribution $< \xi >$ that spans ξ, we can analyse the impact of ξ in this geometry of the vertical distribution leaf by looking at the results that follow.

Theorem 16. *Let $(\bar{O}_1,\phi,\xi,\eta,g_1)$ be a Kenmotsu manifold and $(\bar{O}_2,g_2)$ be a $\mathcal{RM}$ such that $\mathcal{J}$ is a $\mathcal{CBSS}$ from M onto $\bar{O}_2$. Then vertical distribution $(\ker\mathcal{J}_* - <\xi>)$ is defines totally geodesic foliation on M if and only if*

$$\begin{aligned}&\lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{X}_1}\mathcal{J}_*\zeta\bar{U}_1,\mathcal{J}_*\zeta\bar{V}_1)\\ &= (\cos^2\theta_2-\cos^2\theta_1)g_1(\nabla_{\bar{X}_1}\mathfrak{L}\bar{U}_1,\bar{V}_1) + g_1(\mathcal{A}_\omega\bar{V}_1,\zeta\bar{U}_1) - g_1(\mathcal{A}_{\bar{X}_1}\bar{V}_1,\zeta\omega\bar{U}_1)\\ &\quad + g_1(grad\ln\lambda,\bar{X}_1)g_1(\zeta\bar{U}_1,\zeta\bar{V}_1) + g_1(grad\ln\lambda,\zeta\bar{U}_1)g_1(\bar{X}_1,\zeta\bar{V}_1)\\ &\quad - g_1(grad\ln\lambda,\zeta\bar{V}_1)g_1(\bar{X}_1,\zeta\bar{U}_1) - \sin^2\theta_1 g_1([\bar{U}_1,\bar{X}_1],\bar{V}_1),\end{aligned} \tag{39}$$

for $\bar{U}_1,\bar{V}_1 \in \Gamma(\ker\mathcal{J}_*)$ and $\bar{X}_1 \in \Gamma(\ker\mathcal{J}_*)^{\perp}$.

Proof. On taking $\bar{U}_1, \bar{V}_1 \in \Gamma(\ker \mathcal{J}_*)$ and $\bar{X}_1 \in \Gamma(\ker \mathcal{J}_*)^\perp$ with using (12), (13) and (17), we have

$$g_1(\nabla_{\bar{U}_1}\bar{V}_1, \bar{X}_1) = -g_1([\bar{U}_1, \bar{X}_1], \bar{V}_1) + g_1(\nabla_{\bar{X}_1}\phi\omega\bar{U}_1, \bar{V}_1) - g_1(\nabla_{\bar{X}_1}\zeta\bar{U}_1, \phi\bar{V}_1).$$

On using decomposition (16) and Theorem 2, we get

$$\begin{aligned} &g_1(\nabla_{\bar{U}_1}\bar{V}_1, \bar{X}_1) \\ &= -g_1([\bar{U}_1, \bar{X}_1], \bar{V}_1) - \cos^2\theta_1 g_1(\nabla_{\bar{X}_1}\mathfrak{K}\bar{U}_1, \bar{V}_1) - \cos^2\theta_2 g_1(\nabla_{\bar{X}_1}\mathfrak{L}\bar{U}_1, \bar{V}_1) \\ &\quad + g_1(\nabla_{\bar{X}_1}\zeta\omega\bar{U}_1, \bar{V}_1) - g_1(\nabla_{\bar{X}_1}\zeta\bar{U}_1, \omega\bar{V}_1) - g_1(\nabla_{\bar{X}_1}\zeta\bar{U}_1, \zeta\bar{V}_1). \end{aligned}$$

On using equation (7), we can write

$$\begin{aligned} &\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1, \bar{X}_1) \\ &= (\cos^2\theta_2 - \cos^2\theta_1) g_1(\nabla_{\bar{X}_1}\mathfrak{L}\bar{U}_1, \bar{V}_1) - \sin^2\theta_1 g([\bar{U}_1, \bar{X}_1], \bar{V}_1) \\ &\quad + g_1(\mathcal{A}_{\bar{X}_1}\omega\bar{V}_1, \zeta\bar{U}_1) - g_1(\mathcal{A}_{\bar{X}_1}\bar{V}_1, \zeta\omega\bar{U}_1) - g_1(\mathcal{H}\nabla_{\bar{X}_1}\zeta\bar{U}_1, \zeta\bar{V}_1). \end{aligned}$$

Using equation (10), we yields

$$\begin{aligned} &\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1, \bar{X}_1) \\ &= (\cos^2\theta_2 - \cos^2\theta_1) g_1(\nabla_{\bar{X}_1}\mathfrak{L}\bar{U}_1, \bar{V}_1) + g_1(\mathcal{A}_{\bar{X}_1}\omega\bar{V}_1, \zeta\bar{U}_1) \\ &\quad - g_1(\mathcal{A}_{\bar{X}_1}\bar{V}_1, \zeta\omega\bar{U}_1) + \lambda^{-2} g_2((\nabla\mathcal{J}_*)(\bar{X}_1, \zeta\bar{U}_1), \mathcal{J}_*\zeta\bar{V}_1) \\ &\quad - \lambda^{-2} g_2(\nabla^{\mathcal{J}}_{\bar{X}_1}\mathcal{J}_*\zeta\bar{U}_1, \mathcal{J}_*\zeta\bar{V}_1) - \sin^2\theta_1 g_1([\bar{U}_1, \bar{X}_1], \bar{V}_1). \end{aligned}$$

Considering Lemma 1, have

$$\begin{aligned} &\sin^2\theta_1 g_1(\nabla_{\bar{U}_1}\bar{V}_1, \bar{X}_1) \\ &= (\cos^2\theta_2 - \cos^2\theta_1) g_1(\nabla_{\bar{X}_1}\mathfrak{L}\bar{U}_1, \bar{V}_1) + g_1(\mathcal{A}_{\bar{X}_1}\omega\bar{V}_1, \zeta\bar{U}_1) - g_1(\mathcal{A}_{\bar{X}_1}\bar{V}_1, \zeta\omega\bar{U}_1) \\ &\quad + g_1(grad \ln\lambda, \bar{X}_1) g_1(\zeta\bar{U}_1, \zeta\bar{V}_1) + g_1(grad \ln\lambda, \zeta\bar{U}_1) g_1(\bar{X}_1, \zeta\bar{V}_1) \\ &\quad - g_1(grad \ln\lambda, \zeta\bar{V}_1) g_1(\bar{X}_1, \zeta\bar{U}_1) - \lambda^{-2} g_2(\nabla^{\mathcal{J}}_{\bar{X}_1}\mathcal{J}_*\zeta\bar{U}_1, \mathcal{J}_*\zeta\bar{V}_1) \\ &\quad - \sin^2\theta_1 g_1([\bar{U}_1, \bar{X}_1], \bar{V}_1). \end{aligned}$$

This completes the proof of the Theorem.

Similarly, we focused on the totally geodesic prerequisite for horizontal distributions.

Theorem 17. *Let* $(\bar{O}_1, \phi, \xi, \eta, g_1)$ *be a* $\mathcal{SM}$ *and* $(\bar{O}_2, g_2)$ *be a* $\mathcal{RM}$ *such that* $\mathcal{J}$ *is a* $\mathcal{CBSS}$ *from* M *onto* $\bar{O}_2$*. Then horizontal distribution* $(\ker \mathcal{J}_*)^\perp$ *is defines totally* *geodesic foliation on* $\bar{O}_1$ *if and only if*

$$\begin{aligned}
&\lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{X}_1}\mathcal{J}_*\zeta\bar{U}_1,\mathcal{J}_*f\bar{X}_2)-g_1(grad\ln\lambda,\bar{X}_1)g_1(\zeta\bar{U}_1,f\bar{X}_2)\\
&=-g_1(\mathcal{A}_{\bar{X}_1}\zeta\bar{U}_1,t\bar{X}_2)-\eta(\bar{U}_1)g_1(\bar{X}_1,f\bar{X}_2)+\eta(\omega\bar{U}_1)g_1(\bar{X}_1,\bar{X}_2)\\
&\quad+g_1(grad\ln\lambda,\zeta\bar{U}_1)g_1(\bar{X}_1,f\bar{X}_2)-g_1(grad\ln\lambda,f\bar{X}_2)g_1(\bar{X}_1,\zeta\bar{U}_1)\\
&\quad-g_1(grad\ln\lambda,\bar{X}_1)g_1(\zeta\omega\bar{U}_1,\bar{X}_2)-g_1(grad\ln\lambda,\zeta\omega\bar{U}_1)g_1(\bar{X}_1,f\bar{X}_2)\\
&\quad+g_1(grad\ln\lambda,\bar{X}_2)g_1(\bar{X}_1,\zeta\omega\bar{U}_1)+\lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{X}_1}\mathcal{J}_*\zeta\omega\bar{U}_1,\mathcal{J}_*\bar{X}_2),
\end{aligned}\tag{40}$$

for any $\bar{X}_1,\bar{X}_2\in\Gamma(\ker\mathcal{J}_*)^\perp,\bar{U}_1\in\Gamma(\ker J_*)$.

Proof. For any $\bar{X}_1,\bar{X}_2\in\Gamma(\ker\mathcal{J}_*)^\perp$ and $\bar{U}_1\in\Gamma(\ker J_*)$ with using (12), (13) and (17), we have

$$g_1(\nabla_{\bar{X}_1}\bar{X}_2,\bar{U}_1)=g_1(\nabla_{\bar{X}_1}\phi\omega\bar{U}_1,\bar{X}_2)-g_1(\nabla_{\bar{X}_1}\zeta\bar{U}_1,\phi\bar{X}_2)-\eta(\bar{U}_1)g_1(\bar{X}_1,f\bar{X}_2).$$

Taking account the fact from Theorem 2, we can write

$$\begin{aligned}
&\sin^2\theta_1 g_1(\nabla_{\bar{X}_1}\bar{X}_2,\bar{U}_1)\\
&=g_1(\nabla_{\bar{X}_1}\zeta\omega\bar{U}_1,\bar{X}_2)-g_1(\nabla_{\bar{X}_1}\zeta\bar{U}_1,\phi\bar{X}_2)-\eta(\bar{U}_1)g_1(\bar{X}_1,f\bar{X}_2)+\eta(\omega\bar{U}_1)g_1(\bar{X}_1,\bar{X}_2).
\end{aligned}$$

From (7), we get

$$\begin{aligned}
\sin^2\theta_1 g_1(\nabla_{\bar{X}_1}\bar{X}_2,\bar{U}_1)=&-g_1(\mathcal{A}_{\bar{X}_1}\zeta\bar{U}_1,t\bar{X}_2)-\lambda^{-2}g_2(\mathcal{J}_*\nabla_{\bar{X}_1}\zeta\bar{U}_1,\mathcal{J}_*f\bar{X}_2)\\
&+\lambda^{-2}g_2(\mathcal{J}_*\nabla_{\bar{X}_1}\zeta\omega\bar{U}_1,\mathcal{J}_*\bar{X}_2)-\eta(\bar{U}_1)g_1(\bar{X}_1,f\bar{X}_2)\\
&+\eta(\omega\bar{U}_1)g_1(\bar{X}_1,\bar{X}_2).
\end{aligned}$$

Considering Lemma 1, we have

$$\begin{aligned}
&\sin^2\theta_1 g_1(\nabla_{\bar{X}_1}\bar{X}_2,\bar{U}_1)\\
&=-g_1(\mathcal{A}_{\bar{X}_1}\zeta\bar{U}_1,t\bar{X}_2)-\eta(\bar{U}_1)g_1(\bar{X}_1,f\bar{X}_2)+\eta(\omega\bar{U}_1)g_1(\bar{X}_1,\bar{X}_2)\\
&\quad-\lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{X}_1}\mathcal{J}_*\zeta\bar{U}_1,\mathcal{J}_*f\bar{X}_2)+g_1(grad\ln\lambda,\bar{X}_1)g_1(\zeta\bar{U}_1,f\bar{X}_2)\\
&\quad+g_1(grad\ln\lambda,\zeta\bar{U}_1)g_1(\bar{X}_1,f\bar{X}_2)-g_1(grad\ln\lambda,f\bar{X}_2)g_1(\bar{X}_1,\zeta\bar{U}_1)\\
&\quad-g_1(grad\ln\lambda,\bar{X}_1)g_1(\zeta\omega\bar{U}_1,\bar{X}_2)-g_1(grad\ln\lambda,\zeta\omega\bar{U}_1)g_1(\bar{X}_1,f\bar{X}_2)\\
&\quad+g_1(grad\ln\lambda,\bar{X}_2)g_1(\bar{X}_1,\zeta\omega\bar{U}_1)+\lambda^{-2}g_2(\nabla^{\mathcal{J}}_{\bar{X}_1}\mathcal{J}_*\zeta\omega\bar{U}_1,\mathcal{J}_*\bar{X}_2).
\end{aligned}$$

It is now fascinating to investigate if the whole space M can become a locally twisted product manifold under specific circumstances. We find some criteria that make total space M a locally twisted product manifold in the following result.

Theorem 18. *Let $\mathcal{J}$ be a $\mathcal{CBSS}$ from $\mathcal{SM}$ $(\bar{O}_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ $(\bar{O}_2, g_2)$. Then M is a locally twisted product manifold of the form $\bar{O}_{1(\ker \mathcal{J}_*)} \times \bar{O}_{1(\ker \mathcal{J}_*)^\perp}$ if and only if*

$$\begin{aligned}&\frac{1}{\lambda^2} g_2(\nabla^{\mathcal{J}}_{\phi \bar{W}} \mathcal{J}_* \phi \bar{V}, \mathcal{J}_* \phi f \bar{X}) + \eta(\bar{W}) g_1(\bar{V}, t\bar{X}) \\ &= g_1(\phi \bar{V}, \phi \bar{W}) g_1(grad \ln \lambda, \mathcal{J}_* \phi f \bar{V}) - g_1(\nabla_{\bar{V}} \phi \bar{W}, t\bar{X}),\end{aligned} \tag{41}$$

and

$$\begin{aligned}g_1(\bar{X}, \bar{Y}) H = t\mathcal{A}_{\bar{X}} t\bar{Y} - t\bar{X}(\ln \lambda) f\bar{Y} + t(grad \ln \lambda) g_1(\bar{X}, f\bar{Y}) \\ + \phi \mathcal{J}_*(\nabla^{\mathcal{J}}_{\bar{X}} \mathcal{J}_* f\bar{Y}) + g_1(f\bar{X}, \bar{Y})\xi,\end{aligned} \tag{42}$$

where H is mean curvature vector and for any $\bar{V}, \bar{W} \in \Gamma(\ker \mathcal{J}_*)$ and $\bar{X}, \bar{Y} \in \Gamma(\ker \mathcal{J}_*)^\perp$.

Proof. For any $\bar{V}, \bar{W} \in \Gamma(\ker \mathcal{J}_*)$ and $\bar{X}, \bar{Y} \in \Gamma(\ker \mathcal{J}_*)^\perp$, we have

$$g_1(\nabla_{\bar{V}} \bar{W}, \bar{X}) = g_1(\mathcal{H} \nabla_{\bar{V}} \phi \bar{W}, f\bar{X}) + g_1(\mathcal{T}_{\bar{V}} \phi \bar{W}, t\bar{X}) + \eta(\bar{W}) g_1(\bar{V}, t\bar{X}).$$

Since ∇ is torsion free, $[\bar{V}, \phi \bar{W}] \in \Gamma(\ker \mathcal{J}_*)$, we have

$$g_1(\nabla_{\bar{V}} \bar{W}, \bar{X}) = g_1(\nabla_{\bar{V}} \phi \bar{W}, t\bar{X}) + g_1(\nabla_{\phi \bar{W}} \phi \bar{V}, \phi f \bar{X}) + \eta(\bar{W}) g_1(\bar{V}, t\bar{X}).$$

Since $\mathcal{J}$ is $\mathcal{CBSS}$, by using Lemma 1 and from the fact that $g_1(f\bar{X}, \phi \bar{V}) = 0$ for $\bar{X} \in (\ker \mathcal{J}_*)^\perp$and $\bar{V} \in (\ker \mathcal{J}_*)$, we have

$$\begin{aligned}g_1(\nabla_{\bar{V}} \bar{W}, \bar{X}) &= g_1(\nabla_{\bar{V}} \phi \bar{W}, t\bar{X}) + \frac{1}{\lambda^2} g_2(\nabla^{\mathcal{J}}_{\phi \bar{W}} \mathcal{J}_* \phi \bar{V}, \mathcal{J}_*(\phi f \bar{X})) \\ &\quad - g_1(\phi \bar{V}, \phi \bar{W}) g_1(grad \ln \lambda, \mathcal{J}_*(\phi f \bar{V})) + \eta(\bar{W}) g_1(\bar{V}, t\bar{X}).\end{aligned}$$

It follows that $\bar{O}_{1(ker \mathcal{J}_*)}$ is totally geodesic if and only if the equation (41) holds good. On the other hand, for $\bar{X}, \bar{Y} \in \Gamma(\ker \mathcal{J}_*)^\perp, \bar{V} \in \Gamma(\ker \mathcal{J}_*)$, we have

$$\begin{aligned}g_1(\nabla_{\bar{X}} \bar{Y}, \bar{V}) &= g_1(\mathcal{A}_{\bar{X}} t\bar{Y} + \mathcal{V} \nabla_{\bar{X}} t\bar{Y}, \phi \bar{V}) + g_1(\mathcal{A}_{\bar{X}} f\bar{Y} + \mathcal{H} \nabla_{\bar{X}} f\bar{Y}, \phi \bar{V}) \\ &\quad + g_1(f\bar{X}, \bar{Y}) \eta(\bar{V}).\end{aligned}$$

From above equation, we get

$$g_1(\nabla_{\bar{X}} \bar{Y}, \bar{V}) = g_1(\mathcal{A}_{\bar{X}} t\bar{Y}, \phi \bar{V}) + g_1(\mathcal{H} \nabla_{\bar{X}} f\bar{Y}, \phi \bar{V}) + g_1(f\bar{X}, \bar{Y}) \eta(\bar{V}).$$

Since $\mathcal{J}$ is $\mathcal{CBSS}$, on using equation (4) and Lemma 1, we arrived at

$$\begin{aligned} g_1(\nabla_{\bar{X}}\bar{Y},\bar{V}) = & g_1(\mathcal{A}_{\bar{X}}t\bar{Y},\phi\bar{V}) - \frac{1}{\lambda^2}g_2(grad\ln\lambda,\bar{X})\frac{1}{\lambda^2}g_2(\mathcal{J}_*f\bar{Y},\mathcal{J}_*\phi\bar{V}) \\ & - \frac{1}{\lambda^2}g_2(grad\ln\lambda,f\bar{Y})\frac{1}{\lambda^2}g_2(\mathcal{J}_*\bar{X},\mathcal{J}_*\phi\bar{V}) \\ & + \frac{1}{\lambda^2}g_2(\bar{X},f\bar{Y})\frac{1}{\lambda^2}g_2(\mathcal{J}_*grad\ln\lambda,\mathcal{J}_*\phi\bar{V}) \\ & + \frac{1}{\lambda^2}g_2(\nabla^{\mathcal{J}}_{\bar{X}}\mathcal{J}_*f\bar{Y},\mathcal{J}_*\phi\bar{V}) + g_1(f\bar{X},\bar{Y})\eta(\bar{V}). \end{aligned}$$

Moreover using the fact that $g_1(f\bar{X},\phi\bar{V}) = 0$, for $\bar{X} \in \Gamma(\ker\mathcal{J}_*)^\perp, \bar{V} \in \Gamma(\ker\mathcal{J}_*)$, we arrived at

$$\begin{aligned} g_1(\nabla_{\bar{X}}\bar{Y},\bar{V}) = & g_1(\mathcal{A}_{\bar{X}}t\bar{Y},\phi\bar{V}) + \frac{1}{\lambda^2}g_2(\nabla^{\mathcal{J}}_{\bar{X}}\mathcal{J}_*f\bar{Y},\mathcal{J}_*\phi\bar{V}) \\ & - \frac{1}{\lambda^2}g_2(grad\ln\lambda,f\bar{Y})\frac{1}{\lambda^2}g_2(\mathcal{J}_*\bar{X},\mathcal{J}_*\phi\bar{V}) \\ & + \frac{1}{\lambda^2}g_2(\bar{X},f\bar{Y})\frac{1}{\lambda^2}g_2(\mathcal{J}_*grad\ln\lambda,\mathcal{J}_*\phi\bar{V}) + g_1(f\bar{X},\bar{Y})\eta(\bar{V}), \end{aligned}$$

from which, we get the complete result.

5. ϕ-Pluriharmonicity of $\mathcal{CBSS}$

Now, we recall the concept of $\mathcal{J}$-pluriharmonicity which is defined by Y. Ohnita [24] and extend the notion from a almost Hermitian manifold to $\mathcal{ACM}$ manifold.

Let $\mathcal{J}$ be a $\mathcal{CBSS}$ from Kenmotsu manifold $(\bar{O}_1,\phi,\xi,\eta,g_1)$ onto a $\mathcal{RM}$ $(\bar{O}_2,g_2)$ with slant angles θ_1 and θ_2. Then $\mathcal{CBSS}$ is ϕ-pluriharmonic, D_{θ_i}-ϕ-pluriharmonic, $ker\mathcal{J}_*$-ϕ-pluriharmonic, $(ker\mathcal{J}_*)^\perp$-ϕ-pluriharmonic and $((ker\mathcal{J}_*)^\perp - ker\mathcal{J}_*)$-$\phi$-pluriharmonic if

$$(\nabla\mathcal{J}_*)(\bar{W},\bar{Z}) + (\nabla\mathcal{J}_*)(\phi\bar{W},\phi\bar{Z}) = 0, \tag{43}$$

for any $\bar{W},\bar{Z} \in \Gamma(\mathfrak{D}^{\theta_i})$, for any $\bar{W},\bar{Z} \in \Gamma(ker\mathcal{J}_*)$, for any $\bar{W},\bar{Z} \in \Gamma(ker\mathcal{J}_*)^\perp$ and for any $\bar{W} \in \Gamma(ker\mathcal{J}_*)^\perp, \bar{Z} \in \Gamma(ker\mathcal{J}_*)$.

Theorem 19. *Let $\mathcal{J}$ be a $\mathcal{CBSS}$ from $\mathcal{SM}$ $(\bar{O}_1,\phi,\xi,\eta,g_1)$ onto a $\mathcal{RM}$ $(\bar{O}_2,g_2)$ with slant angles θ_1 and θ_2. Suppose that $\mathcal{J}$ is D_{θ_1}-ϕ-pluriharmonic. Then D_{θ_1} defines totally geodesic foliation M_1 if and only if*

$$\begin{aligned} & \mathcal{J}_*(\zeta\mathcal{T}_{\omega\bar{U}}\zeta\omega\bar{V} + f\mathcal{H}\nabla_{\omega\bar{U}}\zeta\omega\bar{V}) - \mathcal{J}_*(\mathcal{A}_{\zeta\bar{U}}\omega\bar{V} + \mathcal{H}\nabla_{\omega\bar{U}}\zeta\bar{V}) \\ & = \cos^2\theta_1\mathcal{J}_*(f\mathcal{T}_{\omega\bar{U}}\bar{V} + \zeta\mathcal{V}\nabla_{\omega\bar{U}}\bar{V}) + \nabla^{\mathcal{J}}_{\omega\bar{U}}\mathcal{J}_*\phi\bar{V} \\ & \quad - \zeta\bar{U}(\ln\lambda)\mathcal{J}_*\zeta\bar{V} - \zeta\bar{V}(\ln\lambda)\mathcal{J}_*\zeta\bar{U} + g_1(\zeta\bar{U},\zeta\bar{V})\mathcal{J}_*(grad\ln\lambda) \end{aligned}$$

for any $\bar{U}, \bar{V} \in \Gamma(D_{\theta_1})$.

Proof. For any $\bar{U}, \bar{V} \in \Gamma(D_{\theta_1})$ and since, $\mathcal{J}$ is D_{θ_1}-ϕ-pluriharmonic, then by using equation (3) and (4), we have

$$\begin{aligned}
0 =& (\nabla \mathcal{J}_*)(\bar{U}, \bar{V}) + (\nabla \mathcal{J}_*)(\phi\bar{U}, \phi\bar{V}) \\
\mathcal{J}_*(\nabla_{\bar{U}}\bar{V}) =& -\mathcal{J}_*(\nabla_{\phi\bar{U}}\phi\bar{V}) + \nabla^{\mathcal{J}}_{\phi\bar{U}}\mathcal{J}_*(\phi\bar{V}) \\
=& \quad -\mathcal{J}_*(\mathcal{A}_{\zeta\bar{U}}\omega\bar{V} + \mathcal{V}\nabla_{\zeta\bar{U}}\omega\bar{V} + \mathcal{T}_{\omega\bar{U}}\zeta\bar{V} + \mathcal{H}\nabla_{\omega\bar{U}}\zeta\bar{V}) + \mathcal{J}_*(\phi\nabla_{\omega\bar{U}}\phi\omega\bar{V} \\
& + (\nabla \mathcal{J}_*)(\zeta\bar{U}, \zeta\bar{V}) - \nabla^{\mathcal{J}}_{\zeta\bar{U}}\mathcal{J}_*\zeta\bar{V} + \nabla^{\mathcal{J}}_{\phi\bar{U}}\mathcal{J}_*\phi\bar{V}
\end{aligned}$$

On using equations (10), (16) with Theorem 2, the above equation finally takes the form

$$\begin{aligned}
\mathcal{J}_*(\nabla_{\bar{U}}V) =& -\cos^2\theta_1\mathcal{J}_*(t\mathcal{T}_{\omega\bar{U}}\bar{V} + f\mathcal{T}_{\omega\bar{U}}\bar{V} + \omega\mathcal{V}\nabla_{\omega\bar{U}}\bar{V} + \zeta\mathcal{V}\nabla_{\omega\bar{U}}\bar{V}) + \nabla^{\mathcal{J}}_{\phi\bar{U}}\mathcal{J}_*\phi\bar{V} \\
& + \mathcal{J}_*(\omega\mathcal{T}_{\omega\bar{U}}\zeta\omega\bar{V} + \zeta\mathcal{T}_{\omega\bar{U}}\zeta\omega\bar{V} + t\mathcal{H}\nabla_{\omega\bar{U}}\zeta\omega\bar{V} + f\mathcal{H}\nabla_{\omega\bar{U}}\zeta\omega\bar{V}) \\
& - \mathcal{J}_*(\mathcal{A}_{\zeta\bar{U}}\omega\bar{V} + \mathcal{V}\nabla_{\zeta\bar{U}}\omega\bar{V} + \mathcal{T}_{\omega\bar{U}}\zeta\bar{V} + \mathcal{H}\nabla_{\omega\bar{U}}\zeta\bar{V}) - \nabla^{\mathcal{J}}_{\zeta\bar{U}}\mathcal{J}_*\zeta\bar{V} \\
& + \zeta\bar{U}(\ln\lambda)\mathcal{J}_*\zeta\bar{V} + \zeta\bar{V}(\ln\lambda)\mathcal{J}_*\zeta\bar{U} - g_1(\zeta\bar{U}, \zeta\bar{V})\mathcal{J}_*(grad\ln\lambda)
\end{aligned}$$

from which we get the desired result.

Theorem 20. *Let $\mathcal{J}$ be a $\mathcal{CBSS}$ from $\mathcal{SM}$ $(\bar{O}_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ $(\bar{O}_2, g_2)$ with slant angles θ_1 and θ_2. Suppose that $\mathcal{J}$ is D_{θ_2}-ϕ-pluriharmonic. Then D_{θ_2} defines totally geodesic foliation M_1 if and only if*

$$\begin{aligned}
& \mathcal{J}_*(\zeta\mathcal{T}_{\omega\bar{Z}}\zeta\omega\bar{W} + f\mathcal{H}\nabla_{\omega\bar{Z}}\zeta\omega\bar{W}) - \mathcal{J}_*(\mathcal{A}_{\zeta\bar{Z}}\omega\bar{W} + \mathcal{H}\nabla_{\omega\bar{Z}}\zeta\bar{W}) \\
&= \cos^2\theta_2\mathcal{J}_*(f\mathcal{T}_{\omega\bar{Z}}\bar{W} + \zeta\mathcal{V}\nabla_{\omega\bar{Z}}\bar{W}) + \nabla^{\mathcal{J}}_{\omega\bar{Z}}\mathcal{J}_*\phi\bar{W} \\
&\quad - \zeta\bar{Z}(\ln\lambda)\mathcal{J}_*\zeta\bar{W} - \zeta\bar{W}(\ln\lambda)\mathcal{J}_*\zeta\bar{Z} + g_1(\zeta\bar{Z}, \zeta\bar{W})\mathcal{J}_*(grad\ln\lambda)
\end{aligned}$$

for any $\bar{Z}, \bar{W} \in \Gamma(D_{\theta_2})$.

Proof. Since, proof of above result is same to Theorem 19. So we omit it.

Theorem 21. *Let $\mathcal{J}$ be a $\mathcal{CBSS}$ from $\mathcal{SM}$ $(\bar{O}_1, \phi, \xi, \eta, g_1)$ onto a $\mathcal{RM}$ $(\bar{O}_2, g_2)$ with slant angles θ_1 and θ_2. Suppose that $\mathcal{J}$ is $((ker\mathcal{J}_*)^\perp - ker\mathcal{J}_*)$-$\phi$-pluriharmonic. Then the horizontal distribution $(ker\mathcal{J}_*)^\perp$ defines totally geodesic foliation on $\bar{O}_1$ if* *and only if*

$$\begin{aligned}
&cos^2\theta_1\mathcal{J}_*\{f\mathcal{T}_{t\bar{X}}\mathfrak{K}\bar{U}+\zeta\mathcal{V}\nabla_{t\bar{X}}\mathfrak{K}\bar{U}+f\mathcal{A}_{C\bar{X}}\mathfrak{K}\bar{U}+\zeta\mathcal{V}\nabla_{C\bar{X}}\mathfrak{K}\bar{U}\}\\
&\quad+cos^2\theta_2\mathcal{J}_*\{f\mathcal{T}_{t\bar{X}}\mathfrak{L}\bar{U}+\zeta\mathcal{V}\nabla_{t\bar{X}}\mathfrak{L}\bar{U}+f\mathcal{A}_{f\bar{X}}\mathfrak{L}U+\zeta\mathcal{V}\nabla_{f\bar{X}}\mathfrak{L}U+\eta(\omega U)f\bar{X}\}\\
&=\mathcal{J}_*\{\zeta\mathcal{T}_{t\bar{X}}\zeta\omega\mathfrak{K}\bar{U}+f\mathcal{H}\nabla_{t\bar{X}}\zeta\omega\mathfrak{K}\bar{U}+\zeta\mathcal{T}_{t\bar{X}}\zeta\omega\mathfrak{L}\bar{U}+f\mathcal{H}\nabla_{t\bar{X}}\zeta\omega\mathfrak{L}\bar{U}\}\\
&\quad+\mathcal{J}_*\{\zeta\mathcal{A}_{f\bar{X}}\zeta\omega\mathfrak{K}\bar{U}+\zeta\mathcal{A}_{f\bar{X}}\zeta\omega\mathfrak{L}\bar{U}-\mathcal{H}\nabla_{t\bar{X}}\zeta\bar{U}\}+\nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U}\\
&\quad-f\bar{X}(\ln\lambda)\mathcal{J}_*\zeta\omega\mathfrak{K}\bar{U}-\zeta\omega\mathfrak{K}\bar{U}(\ln\lambda)\mathcal{J}_*f\bar{X}+g_1(f\bar{X},\zeta\omega\mathfrak{K}\bar{U})\mathcal{J}_*(grad\ln\lambda)\\
&\quad-f\bar{X}(\ln\lambda)\mathcal{J}_*\zeta\omega\mathfrak{L}\bar{U}-\zeta\omega\mathfrak{L}\bar{U}(\ln\lambda)\mathcal{J}_*f\bar{X}+g_1(f\bar{X},\zeta\omega\mathfrak{L}\bar{U})\mathcal{J}_*(grad\ln\lambda)\\
&\quad+\mathcal{J}_*(\nabla_{\bar{X}}\bar{U})+\nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U}+\nabla^{\mathcal{J}}_{f\bar{X}}\mathcal{J}_*\zeta\omega\mathfrak{K}\bar{U}+\nabla^{\mathcal{J}}_{f\bar{X}}\mathcal{J}_*\zeta\omega\mathfrak{L}\bar{U},
\end{aligned}$$

for any $\bar{X}\in\Gamma(ker\mathcal{J}_*)^\perp$ *and* $\bar{U}\in\Gamma(ker\mathcal{J}_*)$

Proof. For any $\bar{X}\in\Gamma(ker\mathcal{J}_*)^\perp$ and $\bar{U}\in\Gamma(ker\mathcal{J}_*)$, since $\mathcal{J}$ is $((ker\mathcal{J}_*)^\perp-ker\mathcal{J}_*)$-$\phi$-pluriharmonic, then by using (4), (10) and (16), we get

$$\mathcal{J}_*(\nabla_{f\bar{X}}\zeta\bar{U})=-\mathcal{J}_*(\nabla_{t\bar{X}}\omega\bar{U}+\nabla_{t\bar{X}}\zeta\bar{U}+\nabla_{f\bar{X}}\omega\bar{U})+\mathcal{J}_*(\nabla_{\bar{X}}\bar{U})+\nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U}.$$

Taking account the fact from (11), we have

$$\begin{aligned}
\mathcal{J}_*(\nabla_{f\bar{X}}\zeta\bar{U})&=-\mathcal{J}_*(\mathcal{T}_{t\bar{X}}\zeta\bar{U}+\mathcal{H}\nabla_{t\bar{X}}\zeta\bar{U})+\mathcal{J}_*(\nabla_{\bar{X}}\bar{U})+\nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U}\\
&\quad+\mathcal{J}_*\{\phi\nabla_{t\bar{X}}\phi\omega\bar{U}-\eta(\nabla_{t\bar{X}}\omega\bar{U})\xi-\eta(\omega\bar{U})t\bar{X}\}\\
&\quad+\mathcal{J}_*\{\phi\nabla_{f\bar{X}}\phi\omega\bar{U}-\eta(\nabla_{f\bar{X}}\omega\bar{U})\xi-\eta(\omega\bar{U})f\bar{X}\}.
\end{aligned}$$

Now on using decomposition (8), Theorem 2 with equations (10), we may yields

$$\begin{aligned}
\mathcal{J}_*(\nabla_{f\bar{X}}\zeta\bar{U})&=\mathcal{J}_*\{-\cos^2\theta_1\phi\nabla_{t\bar{X}}\mathfrak{K}\bar{U}-\cos^2\theta_2\phi\nabla_{t\bar{X}}\mathfrak{L}\bar{U}+\eta(\omega\bar{U})f\bar{X}\}\\
&\quad+\mathcal{J}_*\{\phi\nabla_{t\bar{X}}\zeta\omega\mathfrak{K}\bar{U}+\phi\nabla_{t\bar{X}}\zeta\omega\mathfrak{L}\bar{U}+\phi\nabla_{f\bar{X}}\zeta\omega\mathfrak{K}\bar{U}+\phi\nabla_{f\bar{X}}\zeta\omega\mathfrak{L}\bar{U}\}\\
&\quad+\mathcal{J}_*\{-\cos^2\theta_1\phi\nabla_{f\bar{X}}\mathfrak{K}\bar{U}-\cos^2\theta_2\phi\nabla_{f\bar{X}}\mathfrak{L}\bar{U}\}\\
&\quad-\mathcal{J}_*(\mathcal{H}\nabla_{t\bar{X}}\zeta\bar{U})+\mathcal{J}_*(\nabla_{\bar{X}}\bar{U})+\nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U}.
\end{aligned}$$

From equations (4)-(7) and after simple calculation, we may write

$$\begin{aligned}
&\mathcal{J}_*(\nabla_{f\bar{X}}\zeta\bar{U})\\
&=-cos^2\theta_1\mathcal{J}_*\{f\mathcal{T}_{t\bar{X}}\mathfrak{K}\bar{U}+\zeta\mathcal{V}\nabla_{t\bar{X}}\mathfrak{K}\bar{U}+f\mathcal{A}_{f\bar{X}}\mathfrak{K}\bar{U}+\zeta\mathcal{V}\nabla_{f\bar{X}}\mathfrak{K}\bar{U}\}\\
&\quad-cos^2\theta_2\mathcal{J}_*\{f\mathcal{T}_{t\bar{X}}\mathfrak{L}\bar{U}+\zeta\mathcal{V}\nabla_{t\bar{X}}\mathfrak{L}\bar{U}+f\mathcal{A}_{f\bar{X}}\mathfrak{L}\bar{U}+\zeta\mathcal{V}\nabla_{f\bar{X}}\mathfrak{L}\bar{U}+\eta(\omega\bar{U})f\bar{X}\}\\
&\quad+\mathcal{J}_*\{\zeta\mathcal{T}_{t\bar{X}}\zeta\omega\mathfrak{K}\bar{U}+f\mathcal{H}\nabla_{t\bar{X}}\zeta\omega\mathfrak{K}\bar{U}+\zeta\mathcal{T}_{t\bar{X}}\zeta\omega\mathfrak{L}\bar{U}+f\mathcal{H}\nabla_{t\bar{X}}\zeta\omega\mathfrak{L}\bar{U}\}\\
&\quad+\mathcal{J}_*\{\zeta\mathcal{A}_{f\bar{X}}\zeta\omega\mathfrak{K}\bar{U}+\zeta\mathcal{A}_{f\bar{X}}\zeta\omega\mathfrak{L}\bar{U}-\mathcal{H}\nabla_{t\bar{X}}\zeta\bar{U}\}+\nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U}\\
&\quad+\mathcal{J}_*(f\mathcal{H}\nabla_{f\bar{X}}\zeta\omega\mathfrak{K}\bar{U}+f\mathcal{H}\nabla_{f\bar{X}}\zeta\omega\mathfrak{K}\bar{U})+\mathcal{J}_*(\nabla_{\bar{X}}\bar{U}).
\end{aligned}$$

Since $\mathcal{J}$ is conformal $\mathcal{RS}$, the by using equations (4) and from Lemma 1, we finally have

$$\begin{aligned}
&\mathcal{J}_*(\nabla_{f\bar{X}}\zeta\bar{U})\\
&= -cos^2\theta_1\mathcal{J}_*\{f\mathcal{T}_{t\bar{X}}\mathfrak{K}\bar{U} + \zeta\mathcal{V}\nabla_{t\bar{X}}\mathfrak{K}\bar{U} + f\mathcal{A}_{C\bar{X}}\mathfrak{K}\bar{U} + \zeta\mathcal{V}\nabla_{C\bar{X}}\mathfrak{K}\bar{U}\}\\
&\quad - cos^2\theta_2\mathcal{J}_*\{f\mathcal{T}_{t\bar{X}}\mathfrak{L}\bar{U} + \zeta\mathcal{V}\nabla_{t\bar{X}}\mathfrak{L}\bar{U} + f\mathcal{A}_{f\bar{X}}\mathfrak{L}U + \zeta\mathcal{V}\nabla_{f\bar{X}}\mathfrak{L}U + \eta(\omega U)f\bar{X}\}\\
&\quad + \mathcal{J}_*\{\zeta\mathcal{T}_{t\bar{X}}\zeta\omega\mathfrak{K}\bar{U} + f\mathcal{H}\nabla_{t\bar{X}}\zeta\omega\mathfrak{K}\bar{U} + \zeta\mathcal{T}_{t\bar{X}}\zeta\omega\mathfrak{L}\bar{U} + f\mathcal{H}\nabla_{t\bar{X}}\zeta\omega\mathfrak{L}\bar{U}\}\\
&\quad + \mathcal{J}_*\{\zeta\mathcal{A}_{f\bar{X}}\zeta\omega\mathfrak{K}\bar{U} + \zeta\mathcal{A}_{f\bar{X}}\zeta\omega\mathfrak{L}\bar{U} - \mathcal{H}\nabla_{t\bar{X}}\zeta\bar{U}\} + \nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U}\\
&\quad - f\bar{X}(\ln\lambda)\mathcal{J}_*\zeta\omega\mathfrak{K}\bar{U} - \zeta\omega\mathfrak{K}\bar{U}(\ln\lambda)\mathcal{J}_*f\bar{X} + g_1(f\bar{X},\zeta\omega\mathfrak{K}\bar{U})\mathcal{J}_*(grad\ln\lambda)\\
&\quad - f\bar{X}(\ln\lambda)\mathcal{J}_*\zeta\omega\mathfrak{L}\bar{U} - \zeta\omega\mathfrak{L}\bar{U}(\ln\lambda)\mathcal{J}_*f\bar{X} + g_1(f\bar{X},\zeta\omega\mathfrak{L}\bar{U})\mathcal{J}_*(grad\ln\lambda)\\
&\quad + \mathcal{J}_*(\nabla_{\bar{X}}\bar{U}) + \nabla^{\mathcal{J}}_{\phi\bar{X}}\mathcal{J}_*\zeta\bar{U} + \nabla^{\mathcal{J}}_{f\bar{X}}\mathcal{J}_*\zeta\omega\mathfrak{K}\bar{U} + \nabla^{\mathcal{J}}_{f\bar{X}}\mathcal{J}_*\zeta\omega\mathfrak{L}\bar{U},
\end{aligned}$$

which completes the proof of theorem.

References

[1] M. A. Akyol and Y. Gunduzalp., Hemi-slant submersions from almost product Riemannian manifolds, Gulf J. Math., 4(3) (2016), 15-27.

[2] M. A. Akyol., Conformal semi-slant submersions, International Journal of Geometric Methods in Modern Physics, 14(7) (2017), 1750114.

[3] M. A. Akyol and B. Sahin., Conformal slant submersions, Hacettepe Journal of Mathematics and Statistics, 48(1) (2019), 28-44.

[4] M. A. Akyol and B. Sahin., Conformal anti-invariant submersions from almost Hermitian manifolds, Turkish Journal of Mathematics, 40 (2016), 43-70.

[5] M. A. Akyol and B. Sahin., Conformal semi-invariant submersions, Communications in Contemporary Mathematics, 19 (2017), 1650011.

[6] I. Al-Dayel, T. fatima, S. Deshmukh and M. Shuaib, A note on conformal bi-slant submersion from Kenmotsu manifold, Journal of geometry and physics, 190(2023) 104864.

[7] I. Al-Dayel, M. Shuaib S. Deshmukh and T. fatima., ϕ-pluriharmonicity in quasi bi-slant conformal $\xi^{\perp}$-submersions: a comprehensive study, AIMS-Mathematics, 8(9), (2023), 21746-21768.

[8] P. Baird and J. C. Wood., Harmonic Morphisms Between Riemannian Manifolds, London Mathematical Society Monographs, 29, Oxford University Press, The Clarendon Press. Oxford, (2003).

[9] J.-P. Bourguignon and H. B. Lawson., Jr., Stability and isolation phenomena for YangMills fields, Comm. Math. Phys., 79 (1981), no. 2, 189–230. http://projecteuclid.org/euclid.cmp/1103908963. ¨

[10] D. Chinea., Almost contact metric submersions, Rend. Circ. Mat. Palermo, 34(1) (1985), 89-104.

[11] J. L. Cabrerizo, A. Carriazo, L. M. Fernandez and M. Fernandez., Slant submanifolds in Sasakian manifolds, Glasg. Math. J., 42 (1) (2000), 125-138.

[12] I. K. Erken and C. Murathan., On slant Riemannian submersions for cosymplectic manifolds, Bull. Korean Math. Soc., 51(6) (2014), 1749-1771.

[13] M. Falcitelli, S. Ianus and A. M. Pastore., Riemannian submersions and Related Topics, World Scientific, River Edge, NJ, (2004).

[14] A. Gray., Pseudo-Riemannian almost product manifolds and submersions, J. Math. Mech., 16 (1967), 715–737.

[15] B. Fuglede., Harmonic morphisms between Riemannian manifolds, Annales de l'institut Fourier (Grenoble), 28 (1978), 107-144.

[16] S. Gudmundsson., The geometry of harmonic morphisms, Ph.D. thesis, University of Leeds, (1992).

[17] S. Gudmundsson and J. C. Wood, Harmonic morphisms between almost Hermitian manifolds, Boll. Un. Mat. Ital. B (7) 11 (1997), no. 2, suppl., 185–197.

[18] Y. Gunduzalp., Semi-slant submersions from almost product Riemannian manifolds, Demonstratio Mathematica, 49(3) (2016), 345-356.

[19] S. Ianu¸s and M. Vi¸sinescu., Space-time compaction and Riemannian submersions, In: Rassias, G.(ed.) The Mathematical Heritage of C. F. Gauss, World Scientific, River Edge (1991), 358-371.

[20] S. Ianu¸s and M. Vi¸sinescu., Kaluza-Klein theory with scalar fields and generalised Hopf manifolds, Classical Quantum Gravity, 4 (1987), no. 5, 1317–1325. http://stacks.iop.org/0264-9381/4/ 1317.

[21] T. Ishihara., A mapping of Riemannian manifolds which preserves harmonic functions, Journal of Mathematics of Kyoto University, 19 (1979), 215-229.

[22] Kenmotsu, K. (1972). A class of almost contact Riemannian manifolds. Tohoku Math., 24, 93-103.

[23] M. T. Mustafa., Applications of harmonic morphisms to gravity, J. Math. Phys., 41 (2000), 6918-6929.

[24] Y. Ohnita., On pluriharmonicity of stable harmonic maps, J. London Math. Soc. (2) 2 (1987), 563-568. 2.2

[25] B. O'Neill., The fundamental equations of a submersion, Michigan Math. J., 13 (1966), 459–469. http://projecteuclid.org/euclid.mmj/1028999604.

[26] K. S. Park and R. Prasad., Semi-slant submersions, Bull. Korean Math. Soc. 50(3) (2013), 951-962.

[27] R. Prasad, S. S. Shukla and S. Kumar., On Quasi bi-slant submersions, Mediterr. J. Math., 16 (2019), 155. https://doi.org/10.1007/s00009-019-1434-7.

[28] R. Prasad, M. A. Akyol, P. K. Singh and S. Kumar., On Quasi bi-slant submersions from Kenmotsu manifolds onto any Riemannian manifolds, Journal of Mathematical Extension, 8(16) (2021).

[29] R. Prasad and S. Kumar., Conformal anti-invariant submersions from nearly Kaehler Manifolds, Palestine Journal of Mathematics, 8(2) (2019).

[30] R. Ponge and H. Reckziegel., Twisted products in pseudo-Riemannian geometry, Geom. Dedicata, (1993), 48(1):15-25.

[31] Tanno, S. (1069). The automorphism groups of almost contact metric manfolds. Tohoku Math., J. 21, 21-38.

[32] B. Sahin., Anti-invariant Riemannian submersions from almost Hermitian manifolds, Central European J. Math., 3 (2010), 437-447.

[33] B. Sahin., Semi-invariant Riemannian submersions from almost Hermitian manifolds, Canad. Math. Bull., 56 (2013), 173-183.

[34] B. S¸ahin, Slant submersions from almost Hermitian manifolds, Bull. Math. Soc. Sci. Math. Roumanie. 1 (2011) 93-105.

[35] B.Sahin, M. A. Akyol Conformal Anti-Invariant Submersion From Almost Hermitian Manifolds, Turk J Math (2016) 40: $43-70$

[36] M. Shuaib and T. Fatima. A note on conformal hemi-slant submersions, Afr. Mat. 34, 4 (2023).

[37] 43Sumeet Kumar et al., Conformal hemi-slant submersions from almost hermitian manifolds, Commun. Korean Math. Soc., 35 (2020), No. 3, pp. 999–1018 https://doi.org/10.4134/CKMS.c190448 pISSN: 1225-1763 / eISSN: 2234-3024.

[38] H. M. Tastan, B. S¸ahin and S¸. Yanan, Hemi-slant submersions, Mediterr. J. Math. 13(4) (2016) 2171-2184.

[39] H. Urakawa. (1993). Calculus of Variations and Harmonic Maps. American Mathematical Society, Providence, RI, USA.

[40] B. Watson., Almost Hermitian submersions, J. Differential Geometry, vol. 11, no. 1, pp. (1976), 147–165.

[41] B. Watson., G, G'-Riemannian submersions and nonlinear gauge field equations of general relativity, In: Rassias, T. (ed.) Global Analysis - Analysis on manifolds, dedicated M. Morse. Teubner-Texte Math., 57 (1983), 324-349, Teubner, Leipzig.

Mohammad Shuaib
Department of Mathematics, School of Computer of Science,
Lovely Professional University,
Punjab, India
email: *shuaibyousuf6@gmail.com*

Tanveer Fatima
Department of Mathematics, College of Science,
Taibah University, Yanbu, K.S.A.
email: *tansari@taibahu.edu.sa*