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SEVERAL FINDINGS CONCERNING DIFFERENT SUBCLASSES OF HARMONIC p-VALENT FUNCTIONS LINKED WITH PASCAL DISTRIBUTION SERIES

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Abstract. This research presents outcomes derived from diverse sub-classes of harmonic p-valent functions combined with Pascal Distribution Series, employing a clear and accessible mathematical approach.

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1. Introduction

Suppose that Υ represent the class of all analytic functions f(z), having the series notation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

and normalized by $f'(0) - 1 = 0 = f(0)$ in the open disk $E = \{z \in \mathbb{C} : |z| < 1\}$. Further, let S denote the class of all Normalized Analytic Functions that are univalent in E. (see[5], [8]).
Also, let Θ represent the set of all complex valued harmonic p-valent functions of the form $f_p = h_p + \overline{g}_p$ defined in the open unit disk $E = \{z : |z| < 1\}$, such that

$$h_p(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad and \quad g_p(z) = \sum_{k=0}^{\infty} b_{k+p} z^{k+p} \tag{1.2}$$

are analytic in E.
For f to be sense preserving in E, the condition $|h_p(z)| > |g_p(z)|$ holds *in* E (see [2], [3]).

Let $H \subset \Theta$ contain $f_p = h_p + \overline{g}_p$ such that the functions are harmonic, p-valent

and sense-preserving in E, and normalized by $f_p(0) = f_{p,z}(0) - 1 = 0$. By the sense-preserving property in E, we can easily show that $|b_p| \leq 1$.
Let $H^0 \subset H$ be a subclass such that the functions have the additional property $b_p = 0$.

The function, $f_p \in H$ is said to be Harmonic Starlike of order $\zeta : 0 \leq \zeta < 1$ *in* E if

$$\Re\left(\frac{zf_{p,z}(z) - \overline{z}f_{p,\overline{z}}(z)}{f_p(z)}\right) > \zeta \qquad for\ z \in E \tag{1.3}$$

and represented as class $H^*(\zeta)$
The function, $f_p \in H$ is said to be Harmonic Convex of order $\zeta : 0 \leq \zeta < 1$ *in* E if

$$\Re\left(\frac{z^2 f_{p,zz}(z) + zf_{p,z}(z) + \overline{z}^2 f_{p,\overline{z}\overline{z}}(z) + \overline{z}f_{p,\overline{z}}(z)}{zf_z(z) - \overline{z}f_{p,\overline{z}}}\right) > \zeta \qquad for\ z \in E \tag{1.4}$$

and represented as class $KH(\zeta)$. One may refer to [1],[4],[6] or [10] for particular characteristics of these classes.

Recently, various authors have explored harmonic functions from different viewpoints. However, the exploration of harmonic functions in the context of Probability Distributions is not widely recognized in the literature. Therefore, this current study aims to examine different subclasses of harmonic p-valent functions in connection with Pascal Distribution Series. For recent research on analytic functions related to probability distributions, readers can consult references such as [7] and [9], among others.

Let $\mathfrak{X}$ be a non-negative discrete random variable, with a Pascal Probability Generating Function given as

$$P(\mathfrak{X} = k + p) = \binom{k+p+\delta-1}{\delta-1}\theta^{k+p}(1-\theta)^{\delta} \qquad k \in 0, 1, 2, 3, ...$$

Let $P_\theta^\delta(z)$ be a Power Series whose coefficients are generation from the function above, such that

$$P_\theta^\delta(z) = z^p + \sum_{k=1}^{\infty}\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}z^{k+p} \quad (\delta \geq 1, 0 \leq \theta \leq 1, z \in E) \tag{1.5}$$

It therefore follows that for any $\delta, \sigma \geq 1$ *and* $0 \leq \theta, \phi \leq 1$, we can introduce the operator

$$P_{\theta,\phi}^{\delta,\sigma}(f)(z) = P_\theta^\delta(z) * h_p(z) + \overline{P_\phi^\sigma(z) * g_p(z)} = H_p(Z) + \overline{G_p(z)}$$

where

$$H_p(z) = z^p - \sum_{k=1}^{\infty} \binom{k+p+\delta-2}{\delta-1} \theta^{k+p-1}(1-\theta)^{\delta} a_{k+p} z^{k+p} \tag{1.6}$$

$$G_p(z) = b_p z^p + \sum_{k=1}^{\infty} \binom{k+p+\sigma-2}{\sigma-1} \phi^{k+p-1}(1-\phi)^{\sigma} b_{k+p} z^{k+p}. \tag{1.7}$$

2. Lemmas

Lemma 1. *(See [4]) If $f_p \in KH^0$, with $b_p = 0$, then*

$$|a_{k+p}| \leq \frac{k+p+1}{2}; \quad |b_{k+p}| \leq \frac{k+p-1}{2}.$$

Lemma 2. *(see [6]) Let $f_p = h_p + \overline{g}_p$. If for some $\zeta : 0 \leq \zeta \leq 1$ and the inequality*

$$\sum_{k=1}^{\infty}(k+p-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}|a_{k+p}|$$
$$+\sum_{k=1}^{\infty}(k+p+\zeta)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}|b_{k+p}| \leq (1-\zeta)\theta^{p-1}$$

holds, then f_p is harmonic, sense-preserving, p-valent in E and $f_p \in H^*(\zeta)$.

Lemma 3. *(see [2],[6]) Let $T^n (n = 1, 2)$ consist of functions $f_p = h_p + \overline{g}_p \in H$ and*

$$h_p(z) = z - \sum_{k=1}^{\infty} |a_{k+p}| z^k; \quad g(z) = (-1)^n \sum_{k=0}^{\infty} |b_{k+p}| z^k, |b_p| < 1.$$

Suppose that $TH^*(\zeta) = H^*(\zeta) \cap T^2$ and $TKH(\zeta) = KH(\zeta) \cap T^1$, then $f_p \in TH^*(\zeta)$ provided Lemma 2.2 is satisfied. Also, if $f_p \in TH^*(\zeta)$, then

$$|a_{k+p}| \leq \frac{p-\zeta}{k+p-\zeta}; \quad k \geq 1$$

$$|b_{k+p}| \leq \frac{p-\zeta}{k+p+\zeta}; \quad k \geq 0.$$

Lemma 4. *(see [2],[6]) Let $f_p = h_p + \overline{g}_p$. If for some $\zeta : 0 \leq \zeta \leq 1$ and the inequality*

$$\sum_{k=1}^{\infty}(k+p)(k+p-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}|a_{k+p}|$$
$$+\sum_{k=1}^{\infty}(k+p)(k+p+\zeta)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}|b_{k+p}| \leq (1-\zeta)\theta^{p-1}$$

holds, then f_p is harmonic, sense-preserving, p-valent in E and $f_p \in KH(\zeta)$. In fact, $f_p \in TKH(\zeta)$ (where f_p, h_p and g_p are defined as in Lemma 2.3) provided Lemma 2.4 is satisfied. Also, if $f_p \in TKH(\zeta)$, then

$$|a_{k+p}| \leq \frac{p-\zeta}{(k+p)(k+p-\zeta)}; \quad k \geq 1$$

$$|b_{k+p}| \leq \frac{p-\zeta}{(k+p)(k+p+\zeta)}; \quad k \geq 0.$$

Lemma 5. *(see [2], [6]) If $f_p = h_p + \overline{g}_p \in H^{*,0}$, with $b_p = 0$, then*

$$|a_{k+p}| \leq \frac{(2(k+p)+1)(k+p+1)}{6}; \quad |b_{k+p}| \leq \frac{(2(k+p)-1)(k+p-1)}{6}.$$

3. Results

Theorem 3.1. *Let $\delta, \sigma \geq 1$ and $0 \leq \theta, \phi < 1$. Let $f_p = h_p + \overline{g}_p \in \Theta$. If the inequalities*

$$\sum_{k=1}^{\infty} |a_{k+p}| + \sum_{k=0}^{\infty} |b_{k+p}| \leq 1 \tag{3.1}$$

and

$$p(1-|b_p|) + (1-\theta)^{\delta}\theta^{p-1} + (1-\phi)^{\sigma}\phi^{p-1} \geq \theta^{p-1} + \phi^{p-1} + \frac{\delta\theta^p}{(1-\theta)^p} + \frac{\sigma\phi^p}{(1-\phi)^p} \tag{3.2}$$

holds. Then

$$P_{\theta,\phi}^{\delta,\sigma}(f_p) \in H^*.$$

Proof. To be in H^*, it must be p-valent and sense-preserving, So, it suffices to show that $|H_p'(z)| > |G_p'(z)|$ since $P_{\theta,\phi}^{\delta,\sigma}(f_p) = H_p + \overline{G}_p$

$$H_p(z) = z^p - \sum_{k=1}^{\infty} \binom{k+p+\delta-2}{\delta-1} \theta^{k+p-1}(1-\theta)^{\delta} a_{k+p} z^{k+p}$$

$$H_p'(z) = pz^{p-1} - \sum_{k=1}^{\infty} (k+p) \binom{k+p+\delta-2}{\delta-1} \theta^{k+p-1}(1-\theta)^{\delta} a_{k+p} z^{k+p-1}$$

Also,

$$G_p(z) = b_p z^p + \sum_{k=1}^{\infty} \binom{k+p+\sigma-2}{\sigma-1} \phi^{k+p-1}(1-\phi)^{\sigma} b_{k+p} z^{k+p}$$

$$G_p'(z) = pb_p z^{p-1} + \sum_{k=1}^{\infty}(k+p)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma} b_{k+p} z^{k+p-1}$$

So, since $z \in E$ (i.e. $|z^k| < 1 \quad \forall k$), using 3.1,

$$|H_p'(z)| = p - \sum_{k=1}^{\infty}(k+p)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}$$

$$|G_p'(z)| = p|b_p| + \sum_{k=1}^{\infty}(k+p)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}$$

We now compute the sense-preserving property

$$\begin{aligned}
|H_p'(z)| - |G_p'(z)| &> p - \sum_{k=1}^{\infty}(k+p)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta} \\
&\quad - p|b_p| - \sum_{k=1}^{\infty}(k+p)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma} \\
&= p(1-|b_p|) - \sum_{k=1}^{\infty}(k+p-1+1)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta} \\
&\quad - \sum_{k=1}^{\infty}(k+p-1+1)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma} \\
&= p(1-|b_p|) - (1-\theta)^{\delta}\sum_{k=1}^{\infty}(k+p-1)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1} \\
&\quad - (1-\theta)^{\delta}\sum_{k=1}^{\infty}\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1} \\
&\quad - (1-\phi)^{\sigma}\sum_{k=1}^{\infty}(k+p-1)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1} \\
&\quad - (1-\phi)^{\sigma}\sum_{k=1}^{\infty}\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}
\end{aligned}$$

Clearly,

$$(k+p-1)\binom{k+p+\delta-2}{\delta-1} = \delta\binom{k+p+\delta-2}{\delta}$$
$$(k+p-1)\binom{k+p+\sigma-2}{\sigma-1} = \sigma\binom{k+p+\sigma-2}{\sigma}$$

Thus,

$$\begin{aligned}|H_p'(z)| - |G_p'(z)| > p(1-|b_p|) &- \delta\theta(1-\theta)^{\delta}\sum_{k=1}^{\infty}\binom{k+p+\delta-2}{\delta}\theta^{k+p-2}\\ &- (1-\theta)^{\delta}\sum_{k=1}^{\infty}\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}\\ &- \sigma\phi(1-\phi)^{\sigma}\sum_{k=1}^{\infty}\binom{k+p+\sigma-2}{\sigma}\phi^{k+p-2}\\ &- (1-\phi)^{\sigma}\sum_{k=1}^{\infty}\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}\end{aligned}$$

$$\begin{aligned}|H_p'(z)| - |G_p'(z)| &> p(1-|b_p|) - \delta\theta(1-\theta)^{\delta}\sum_{k=0}^{\infty}\binom{k+p+\delta-1}{\delta}\theta^{k+p-1}\\ &- (1-\theta)^{\delta}\sum_{k=0}^{\infty}\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}\\ &+ (1-\theta)^{\delta}\binom{p+\delta-2}{\delta-1}\theta^{p-1} + (1-\phi)^{\sigma}\binom{p+\sigma-2}{\sigma-1}\phi^{p-1}\\ &- \sigma\phi(1-\phi)^{\sigma}\sum_{k=0}^{\infty}\binom{k+p+\sigma-1}{\sigma}\phi^{k+p-1}\\ &- (1-\phi)^{\sigma}\sum_{k=0}^{\infty}\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}\\ &= p(1-|b_p|) - \delta\theta(1-\theta)^{\delta}\frac{\theta^{p-1}}{(1-\theta)^{p+\delta}} - (1-\theta)^{\delta}\frac{\theta^{p-1}}{(1-\theta)^{\delta}}\\ &+ (1-\theta)^{\delta}\theta^{p-1} + (1-\phi)^{\sigma}\phi^{p-1} - \sigma\phi(1-\phi)^{\sigma}\frac{\phi^{p-1}}{(1-\phi)^{p+\sigma}}\\ &- (1-\phi)^{\sigma}\frac{\phi^{p-1}}{(1-\phi)^{\sigma}}\\ &= p(1-|b_p|) - \frac{\delta\theta^{p}}{(1-\theta)^{p}} - \theta^{p-1} + (1-\theta)^{\delta}\theta^{p-1} + (1-\phi)^{\sigma}\phi^{p-1}\\ &- \frac{\sigma\phi^{p}}{(1-\phi)^{p}} - \phi^{p-1} \geq 0 \quad (by \;\; 3.2)\end{aligned}$$

Thus, we have confirmed $P_{\theta,\phi}^{\delta,\sigma}$ is indeed sense-preserving.

Indeed, $P_{\theta,\phi}^{\delta,\sigma}(f_p)$ is p-valent in E. By 3.1, we have that

$$\Re(|H_p'(z(t))|) - |G_p'(z(t))| > p - \sum_{k=1}^{\infty}(k+p)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}$$
$$- p|b_p| - \sum_{k=1}^{\infty}(k+p)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}$$

Using 3.2, we see that the inequality above is non-negative. Thus

$$\Re\left(\frac{P_{\theta,\phi}^{\delta,\sigma}(f_p)(z_2) - P_{\theta,\phi}^{\delta,\sigma}(f_p)(z_1)}{z_2 - z_1}\right) > \int(\Re(|H_p'(z(t))|) - |G_p'(z(t))|)dt \quad > 0$$

Now, to show that $P_{\theta,\phi}^{\delta,\sigma}(f_p) \in SH^*$, we verify that $\psi_1 \leq p$, where

$$\psi_1 = \sum_{k=1}^{\infty}(k+p)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}|a_{k+p}| + p|b_p|$$
$$+ \sum_{k=1}^{\infty}(k+p)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}|b_{k+p}|$$

Since $|a_{k+p}| \le 1, |b_{k+p}| \le 1, \quad \forall n \ge 1$, Using 3.1 and 3.2,

$$\begin{aligned}
\psi_1 &= p|b_p| + \delta\theta(1-\theta)^{\delta} \sum_{k=0}^{\infty} \binom{k+p+\delta-1}{\delta} \theta^{k+p-1} \\
&\quad + (1-\theta)^{\delta} \sum_{k=0}^{\infty} \binom{k+p+\delta-2}{\delta-1} \theta^{k+p-1} \\
&\quad - (1-\theta)^{\delta} \binom{p+\delta-2}{\delta-1} \theta^{p-1} - (1-\phi)^{\sigma} \binom{p+\sigma-2}{\sigma-1} \phi^{p-1} \\
&\quad + \sigma\phi(1-\phi)^{\sigma} \sum_{k=0}^{\infty} \binom{k+p+\sigma-1}{\sigma} \phi^{k+p-1} \\
&\quad + (1-\phi)^{\sigma} \sum_{k=0}^{\infty} \binom{k+p+\sigma-2}{\sigma-1} \phi^{k+p-1} \\
&= p|b_p| + \delta\theta(1-\theta)^{\delta} \frac{\theta^{p-1}}{(1-\theta)^{p+\delta}} + (1-\theta)^{\delta} \frac{\theta^{p-1}}{(1-\theta)^{\delta}} \\
&\quad - (1-\theta)^{\delta}\theta^{p-1} - (1-\phi)^{\sigma}\phi^{p-1} + \sigma\phi(1-\phi)^{\sigma} \frac{\phi^{p-1}}{(1-\phi)^{p+\sigma}} \\
&\quad + (1-\phi)^{\sigma} \frac{\phi^{p-1}}{(1-\phi)^{\sigma}} \\
&= p|b_p| + \frac{\delta\theta^p}{(1-\theta)^p} + \theta^{p-1} - (1-\theta)^{\delta}\theta^{p-1} - (1-\phi)^{\sigma}\phi^{p-1} + \frac{\sigma\phi^p}{(1-\phi)^p} + \phi^{p-1} \\
&\le p
\end{aligned}$$

Thus, $P_{\theta,\phi}^{\delta,\sigma}(f_p) \in H^*$.

It will be interesting to note that for specific values of p, θ and ϕ, the following corollaries can be deduced.

Corollary 3.2. *Let $\delta \ge 1$, $\sigma \ge 1$ and $0 \le \theta < 1$, $0 \le \phi < 1$. Let $f_p = h_p + \overline{g}_p \in \Omega$. If $p = 1$ and the inequalities*

$$\sum_{k=1}^{\infty} |a_{k+1}| + \sum_{k=0}^{\infty} |b_{k+1}| \le 1$$

and

$$1 - |b_1| + (1-\theta)^{\delta} + (1-\phi)^{\sigma} \ge 2 + \frac{\delta\theta}{(1-\theta)} + \frac{\sigma\phi}{(1-\phi)}$$

holds. Then

$$P_{\theta,\phi}^{\delta,\sigma}(f_1) \in H^*.$$

Corollary 3.3. *Let $\sigma \geq 1$ and $0 \leq \phi < 1$. Let $f_p = h_p + \overline{g}_p \in \Omega$. If $\delta = 1, \theta = 0, p = 1$ and the inequalities*

$$\sum_{k=1}^{\infty} |a_{k+1}| + \sum_{k=0}^{\infty} |b_{k+1}| \leq 1$$

and

$$2 - |b_1| + (1-\phi)^{\sigma} \geq 2 + \frac{\sigma\phi}{(1-\phi)}$$

holds. Then

$$P_{0,\phi}^{1,\sigma}(f_1) \in H^*.$$

Corollary 3.4. *Let $f_p = h_p + \overline{g}_p \in \Omega$. If $\delta = 1, \sigma = 1\theta = 0, \phi = 0, p = 1$ and the inequalities*

$$\sum_{k=1}^{\infty} |a_{k+1}| + \sum_{k=0}^{\infty} |b_{k+1}| \leq 1$$

and

$$3 - |b_1| \geq 2$$

holds. Then

$$P_{0,0}^{1,1}(f_1) \in H^*.$$

Theorem 3.5. *Let $0 \leq \zeta < 1$, $\delta, \sigma \geq 1$ and $0 \leq \theta, \phi < 1$. If the inequality*

$$\frac{\delta(\delta+1)\theta^{p+1}}{(1-\theta)^{p+1}} + \frac{(4-\zeta)\delta\theta^{p}}{(1-\theta)^{p}} + \frac{\sigma(\sigma+1)\phi^{p+1}}{(1-\phi)^{p+1}} + \frac{(2+\zeta)\sigma\phi^{p}}{(1-\phi)^{p}} \leq 2(1-\zeta)(1-\theta)^{\delta}\binom{p+\delta-2}{\delta-1}\theta^{p-1} \tag{3.3}$$

holds. Then

$$P_{\theta,\phi}^{\delta,\sigma}(KH^0) \subset H^{*,0}(\zeta).$$

Proof. If $f_p \in KH^0$, then it suffices to show that $P_{\theta,\phi}^{\delta,\sigma}(KH^0) \subset H^{*,0}(\zeta)$. In essence, we must show that $\psi_2 \leq (1-\zeta)\theta^{p-1}$, where

$$\psi_2 = \sum_{k=1}^{\infty}(k+p-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}|a_{k+p}| + \sum_{k=1}^{\infty}(k+p+\zeta)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}|b_{k+p}|$$

Using Lemma 2.1,

$$\begin{aligned}
\psi_2 \leq \frac{1}{2}\Bigg[&\sum_{k=1}^{\infty}(k+p-\zeta)(k+p+1)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}\\
&+\sum_{k=1}^{\infty}(k+p+\zeta)(k+p-1)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}\Bigg]\\
=\frac{1}{2}\Bigg[&\sum_{k=1}^{\infty}[(k+p-1)(k+p-2)+(4-\zeta)(k+p-1)\\
&+2(1-\zeta)]\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}\\
&+\sum_{k=1}^{\infty}[(k+p-1)(k+p-2)+(2+\zeta)(k+p-1)]\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}\Bigg]\\
=\frac{1}{2}\Bigg[&\sum_{k=1}^{\infty}(k+p-1)(k+p-2)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}\\
&+\sum_{k=1}^{\infty}(4-\zeta)(k+p-1)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}\\
&+\sum_{k=1}^{\infty}2(1-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}\\
&+\sum_{k=1}^{\infty}(k+p-1)(k+p-2)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}\\
&+\sum_{k=1}^{\infty}(2+\zeta)(k+p-1)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}\Bigg]\\
=\frac{1}{2}\Bigg[&\sum_{k=1}^{\infty}\delta(\delta+1)\binom{k+p+\delta-2}{\delta+1}\theta^{k+p-1}(1-\theta)^{\delta}\\
&+\sum_{k=1}^{\infty}(4-\zeta)\delta\binom{k+p+\delta-2}{\delta}\theta^{k+p-1}(1-\theta)^{\delta}\\
&+\sum_{k=1}^{\infty}2(1-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}\\
&+\sum_{k=1}^{\infty}\sigma(\sigma+1)\binom{k+p+\sigma-2}{\sigma+1}\phi^{k+p-1}(1-\phi)^{\sigma}\\
&+\sum_{k=1}^{\infty}(2+\zeta)\sigma\binom{k+p+\sigma-2}{\sigma}\phi^{k+p-1}(1-\phi)^{\sigma}\Bigg]
\end{aligned}$$

$$\begin{aligned}
\psi_2 \leq \frac{1}{2}\Bigg[& \delta(\delta+1)\theta^2(1-\theta)^\delta \sum_{k=0}^{\infty} \binom{k+p+\delta}{\delta+1} \theta^{k+p-1} \\
& + (4-\zeta)\delta\theta(1-\theta)^\delta \sum_{k=0}^{\infty} \binom{k+p+\delta-1}{\delta} \theta^{k+p-1} \\
& + 2(1-\zeta)(1-\theta)^\delta \sum_{k=1}^{\infty} \binom{k+p+\delta-2}{\delta-1} \theta^{k+p-1} \\
& + \sigma(\sigma+1)\phi^2(1-\phi)^\sigma \sum_{k=0}^{\infty} \binom{k+p+\sigma}{\sigma+1} \phi^{k+p-1} \\
& + (2+\zeta)\sigma\phi(1-\phi)^\sigma \sum_{k=0}^{\infty} \binom{k+p+\sigma-1}{\sigma} \phi^{k+p-1} \Bigg] \\
= \frac{1}{2}\Bigg[& \delta(\delta+1)\theta^2(1-\theta)^\delta \frac{\theta^{p-1}}{(1-\theta)^{p+\delta+1}} + (4-\zeta)\delta\theta(1-\theta)^\delta \frac{\theta^{p-1}}{(1-\theta)^{p+\delta}} \\
& + 2(1-\zeta)(1-\theta)^\delta \frac{\theta^{p-1}}{(1-\theta)^\delta} + \sigma(\sigma+1)\phi^2(1-\phi)^\sigma \frac{\phi^{p-1}}{(1-\phi)^{p+\sigma+1}} \\
& + (2+\zeta)\sigma\phi(1-\phi)^\sigma \frac{\phi^{p-1}}{(1-\phi)^{p+\sigma}} - 2(1-\zeta)(1-\theta)^\delta \binom{p+\delta-2}{\delta-1} \theta^{p-1} \Bigg] \\
= \frac{1}{2}\Bigg[& \frac{\delta(\delta+1)\theta^{p+1}}{(1-\theta)^{p+1}} + \frac{(4-\zeta)\delta\theta^p}{(1-\theta)^p} + 2(1-\zeta)\theta^{p-1} \\
& + \frac{\sigma(\sigma+1)\phi^{p+1}}{(1-\phi)^{p+1}} + \frac{(2+\zeta)\sigma\phi^p}{(1-\phi)^p} - 2(1-\zeta)(1-\theta)^\delta \binom{p+\delta-2}{\delta-1} \theta^{p-1} \Bigg]
\end{aligned}$$

Using 3.3,

$$\begin{aligned}
\psi_2 \leq \frac{1}{2}[& 2(1-\zeta)(1-\theta)^\delta \binom{p+\delta-2}{\delta-1} \theta^{p-1} + 2(1-\zeta)\theta^{p-1} \\
& - 2(1-\zeta)(1-\theta)^\delta \binom{p+\delta-2}{\delta-1} \theta^{p-1}] \\
= \frac{1}{2}[& 2(1-\zeta)\theta^{p-1}]
\end{aligned}$$

So, $\psi_2 \leq (1-\zeta)\theta^{p-1}$ and $P_{\theta,\phi}^{\delta,\sigma}(KH^0) \subset H^{*,0}(\zeta)$.

Theorem 3.6. *Let $0 \leq \zeta < 1$, $\delta, \sigma \geq 1$ and $0 \leq \theta, \phi < 1$. If the inequality*

$$\begin{aligned} &\frac{2\delta(\delta+1)(\delta+2)\theta^{p+2}}{(1-\theta)^{p+2}} + \frac{(15-2\zeta)\delta(\delta+1)\theta^{p+1}}{(1-\theta)^{p+1}} + \frac{(24-9\zeta)\delta\theta^{p}}{(1-\theta)^{p}} \\ &+ \frac{2\sigma(\sigma+1)(\sigma+2)\phi^{p+2}}{(1-\phi)^{p+2}} + \frac{(9+2\zeta)\sigma(\sigma+1)\phi^{p+1}}{(1-\phi)^{p+1}} + \frac{(6+3\zeta)\sigma\phi^{p}}{(1-\phi)^{p}} \\ &\qquad\qquad \leq 6(1-\zeta)(1-\theta)^{\delta}\binom{p+\delta-2}{\delta-1}\theta^{p-1} \end{aligned} \tag{3.4}$$

holds. Then

$$P^{\delta,\sigma}_{\theta,\phi}(SH^{*,0}(\zeta)) \subset H^{*,0}(\zeta).$$

Proof. Again, it suffices to show that $P^{\delta,\sigma}_{\theta,\phi}(H^{*,0}(\zeta)) \subset H^{*,0}(\zeta)$, by showing that $\psi_2 \leq (1-\zeta)\theta^{p-1}$ where

$$\begin{aligned} \psi_2 = &\sum_{k=1}^{\infty}(k+p-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}|a_{k+p}| \\ &+ \sum_{k=1}^{\infty}(k+p+\zeta)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}|b_{k+p}| \end{aligned}$$

Using Lemma 2.5,

$$\begin{aligned}
\psi_2 \leq \frac{1}{6}\Bigg[&\sum_{k=1}^{\infty}(2(k+p)+1)(k+p+1)(k+p-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}\\
&+\sum_{k=1}^{\infty}(2(k+p)-1)(k+p-1)(k+p+\zeta)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}\Bigg]\\
= \frac{1}{6}\Bigg[&\sum_{k=1}^{\infty}2(k+p-1)(k+p-2)(k+p-3)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}\\
&+\sum_{k=1}^{\infty}(15-2\zeta)(k+p-1)(k+p-2)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}\\
&+\sum_{k=1}^{\infty}(24-9\zeta)(k+p-1)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}\\
&+\sum_{k=1}^{\infty}6(1-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}\\
&+\sum_{k=1}^{\infty}2(k+p-1)(k+p-2)(k+p-3)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}\\
&+\sum_{k=1}^{\infty}(9+2\zeta)(k+p-1)(k+p-2)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}\\
&+\sum_{k=1}^{\infty}(6+3\zeta)(k+p-1)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}\Bigg]
\end{aligned}$$

$$\psi_2 \leq \frac{1}{6}\Bigg[\sum_{k=1}^{\infty} 2\delta(\delta+1)(\delta+2)\binom{k+p+\delta-2}{\delta+2}\theta^{k+p-1}(1-\theta)^{\delta}$$
$$+\sum_{k=1}^{\infty}(15-2\zeta)\delta(\delta+1)\binom{k+p+\delta-2}{\delta+1}\theta^{k+p-1}(1-\theta)^{\delta}$$
$$+\sum_{k=1}^{\infty}(24-9\zeta)\delta\binom{k+p+\delta-2}{\delta}\theta^{k+p-1}(1-\theta)^{\delta}$$
$$+\sum_{k=1}^{\infty}6(1-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}$$
$$+\sum_{k=1}^{\infty}2\sigma(\sigma+1)(\sigma+2)\binom{k+p+\sigma-2}{\sigma+2}\phi^{k+p-1}(1-\phi)^{\sigma}$$
$$+\sum_{k=1}^{\infty}(9+2\zeta)\sigma(\sigma+1)\binom{k+p+\sigma-2}{\sigma+1}\phi^{k+p-1}(1-\phi)^{\sigma}$$
$$+\sum_{k=1}^{\infty}(6+3\zeta)\sigma\binom{k+p+\sigma-2}{\sigma}\phi^{k+p-1}(1-\phi)^{\sigma}\Bigg]$$
$$=\frac{1}{6}\Bigg[2\delta(\delta+1)(\delta+2)\theta^{3}(1-\theta)^{\delta}\sum_{k=0}^{\infty}\binom{k+p+\delta+1}{\delta+2}\theta^{k+p-1}$$
$$+(15-2\zeta)\delta(\delta+1)\theta^{2}(1-\theta)^{\delta}\sum_{k=0}^{\infty}\binom{k+p+\delta}{\delta+1}\theta^{k+p-1}$$
$$+(24-9\zeta)\delta\theta(1-\theta)^{\delta}\sum_{k=0}^{\infty}\binom{k+p+\delta-1}{\delta}\theta^{k+p-1}$$
$$+6(1-\zeta)(1-\theta)^{\delta}\sum_{k=1}^{\infty}\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}$$
$$+2\sigma(\sigma+1)(\sigma+2)\phi^{3}(1-\phi)^{\sigma}\sum_{k=0}^{\infty}\binom{k+p+\sigma+1}{\sigma+2}\phi^{k+p-1}$$
$$+(9+2\zeta)\sigma(\sigma+1)\phi^{2}(1-\phi)^{\sigma}\sum_{k=0}^{\infty}\binom{k+p+\sigma}{\sigma+1}\phi^{k+p-1}$$
$$+(6+3\zeta)\sigma\phi(1-\phi)^{\sigma}\sum_{k=0}^{\infty}\binom{k+p+\sigma-1}{\sigma}\phi^{k+p-1}\Bigg]$$

$$\begin{aligned}\psi_2 \le \frac{1}{6}\Bigg[&2\delta(\delta+1)(\delta+2)\theta^3(1-\theta)^\delta \frac{\theta^{p-1}}{(1-\theta)^{p+\delta+2}}\\
&+(15-2\zeta)\delta(\delta+1)\theta^2(1-\theta)^\delta \frac{\theta^{p-1}}{(1-\theta)^{p+\delta+1}}\\
&+(24-9\zeta)\delta\theta(1-\theta)^\delta \frac{\theta^{p-1}}{(1-\theta)^{p+\delta}}+6(1-\zeta)(1-\theta)^\delta\frac{\theta^{p-1}}{(1-\theta)^\delta}\\
&+2\sigma(\sigma+1)(\sigma+2)\phi^3(1-\phi)^\sigma \frac{\phi^{p-1}}{(1-\phi)^{p+\sigma+2}}\\
&+(9+2\zeta)\sigma(\sigma+1)\phi^2(1-\phi)^\sigma \frac{\phi^{p-1}}{(1-\phi)^{p+\sigma+1}}\\
&+(6+3\zeta)\sigma\phi(1-\phi)^\sigma \frac{\phi^{p-1}}{(1-\phi)^{p+\sigma}}-6(1-\zeta)(1-\theta)^\delta\binom{p+\delta-2}{\delta-1}\theta^{p-1}\Bigg]\\
=\frac{1}{6}\Bigg[&\frac{2\delta(\delta+1)(\delta+2)\theta^{p+2}}{(1-\theta)^{p+2}}+\frac{(15-2\zeta)\delta(\delta+1)\theta^{p+1}}{(1-\theta)^{p+1}}+\frac{(24-9\zeta)\delta\theta^p}{(1-\theta)^p}\\
&+6(1-\zeta)\theta^{p-1}+\frac{2\sigma(\sigma+1)(\sigma+2)\phi^{p+2}}{(1-\phi)^{p+2}}+\frac{(9+2\zeta)\sigma(\sigma+1)\phi^{p+1}}{(1-\phi)^{p+1}}\\
&+\frac{(6+3\zeta)\sigma\phi^p}{(1-\phi)^p}-6(1-\zeta)(1-\theta)^\delta\binom{p+\delta-2}{\delta-1}\theta^{p-1}\Bigg]\end{aligned}$$

Using 3.4,

$$\begin{aligned}\psi_2 &\le \frac{1}{6}[6(1-\zeta)(1-\theta)^\delta\binom{p+\delta-2}{\delta-1}\theta^{p-1}+6(1-\zeta)\theta^{p-1}\\
&\quad-6(1-\zeta)(1-\theta)^\delta\binom{p+\delta-2}{\delta-1}\theta^{p-1}]\\
&=\frac{1}{6}[6(1-\zeta)\theta^{p-1}]\end{aligned}$$

So, $\psi_2 \le (1-\zeta)\theta^{p-1}$ and $P^{\delta,\sigma}_{\theta,\phi}(H^{*,0}(\zeta)) \subset H^{*,0}(\zeta)$.

Theorem 3.7. *Let $0 \le \zeta < 1$, $\delta, \sigma \ge 1$ and $0 \le \theta, \phi < 1$. If the inequality*

$$\binom{p+\sigma-2}{\sigma-1}(p+\zeta)|b_p|\phi^{p-1}+(p-\zeta)\phi^{p-1} \le (p-\zeta)\Bigg[(1-\theta)^\delta\binom{p+\delta-2}{\delta-1}\theta^{p-1} +(1-\phi)^\sigma\binom{p+\sigma-2}{\sigma-1}\phi^{p-1}\Bigg] \tag{3.5}$$

holds. Then

$$P^{\delta,\sigma}_{\theta,\phi}(TH^*(\zeta) \subset TH^*(\zeta).$$

Proof. It suffices to show that $P_{\theta,\phi}^{\delta,\sigma}(TH^*(\zeta) \subset TH^*(\zeta)$. In essence, we must show that $\psi_3 \leq (p-\zeta)\theta^{p-1}$, where

$$\psi_3 = \sum_{k=1}^{\infty}(k+p-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}|a_{k+p}|$$
$$+\sum_{k=1}^{\infty}(k+p+\zeta)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}|b_{k+p}|$$
$$+(p+\zeta)\binom{p+\sigma-2}{\sigma-1}|b_p|\phi^{p-1}$$

Since $f_p \in TH^*(\zeta)$, then

$$|a_{k+p}| \leq \frac{p-\zeta}{k+p-\zeta}, k \geq 1 \quad and \quad |b_{k+p}| \leq \frac{p-\zeta}{k+p+\zeta}, k \geq 0$$

$$\psi_3 \leq \sum_{k=1}^{\infty}(p-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}$$
$$+\sum_{k=1}^{\infty}(p-\zeta)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}$$
$$+(p+\zeta)\binom{p+\sigma-2}{\sigma-1}|b_p|\phi^{p-1}$$
$$=(p-\zeta)\left[(1-\theta)^{\delta}\sum_{k=1}^{\infty}\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}+(1-\phi)^{\sigma}\sum_{k=1}^{\infty}\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}\right]$$
$$+(p+\zeta)\binom{p+\sigma-2}{\sigma-1}|b_p|\phi^{p-1}$$
$$=(p-\zeta)\left[(1-\theta)^{\delta}\sum_{k=0}^{\infty}\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}-(1-\theta)^{\delta}\binom{p+\delta-2}{\delta-1}\theta^{p-1}\right.$$
$$\left.+(1-\phi)^{\sigma}\sum_{k=0}^{\infty}\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}-(1-\phi)^{\sigma}\binom{p+\sigma-2}{\sigma-1}\phi^{p-1}\right]$$
$$+(p+\zeta)\binom{p+\sigma-2}{\sigma-1}|b_p|\phi^{p-1}$$
$$=(p-\zeta)\left[(1-\theta)^{\delta}\frac{\theta^{p-1}}{(1-\theta)^{\delta}}-(1-\theta)^{\delta}\binom{p+\delta-2}{\delta-1}\theta^{p-1}+(1-\phi)^{\sigma}\frac{\phi^{p-1}}{(1-\phi)^{\sigma}}\right.$$
$$\left.-(1-\phi)^{\sigma}\binom{p+\sigma-2}{\sigma-1}\phi^{p-1}\right]+(p+\zeta)\binom{p+\sigma-2}{\sigma-1}|b_p|\phi^{p-1}$$

$$\begin{aligned}\psi_3 &\leq (p-\zeta)\theta^{p-1} - (p-\zeta)(1-\theta)^{\delta}\binom{p+\delta-2}{\delta-1}\theta^{p-1} + (p-\zeta)\phi^{p-1}\\ &\quad - (p-\zeta)(1-\phi)^{\sigma}\binom{p+\sigma-2}{\sigma-1}\phi^{p-1} + (p+\zeta)\binom{p+\sigma-2}{\sigma-1}|b_p|\phi^{p-1}\\ &= (p-\zeta)\theta^{p-1} + (p-\zeta)\phi^{p-1} + (p+\zeta)\binom{p+\sigma-2}{\sigma-1}|b_p|\phi^{p-1}\\ &\quad - (p-\zeta)\left[(1-\theta)^{\delta}\binom{p+\delta-2}{\delta-1}\theta^{p-1} + (1-\phi)^{\sigma}\binom{p+\sigma-2}{\sigma-1}\phi^{p-1}\right]\end{aligned}$$

Using 3.5,

$$\begin{aligned}\psi_3 &\leq (p-\zeta)\theta^{p-1} + (p-\zeta)\phi^{p-1} + (p+\zeta)\binom{p+\sigma-2}{\sigma-1}|b_p|\phi^{p-1}\\ &\quad - (p-\zeta)\left[(1-\theta)^{\delta}\binom{p+\delta-2}{\delta-1}\theta^{p-1} + (1-\phi)^{\sigma}\binom{p+\sigma-2}{\sigma-1}\phi^{p-1}\right]\\ &= (p-\zeta)\theta^{p-1}\end{aligned}$$

So, $\psi_3 \leq (p-\zeta)\theta^{p-1}$ and $P_{\theta,\phi}^{\delta,\sigma}(TH^*(\zeta) \subset TH^*(\zeta)$.

Theorem 3.8. *Let* $0 \leq \zeta < 1$, $\delta, \sigma \geq 1$ *and* $0 \leq \theta, \phi < 1$. *If the inequality*

$$\begin{aligned}&\frac{\delta(\delta+1)(\delta+2)\theta^{p+2}}{(1-\theta)^{p+2}} + \frac{(7-\zeta)\delta(\delta+1)\theta^{p+1}}{(1-\theta)^{p+1}} + \frac{(10-4\zeta)\delta\theta^{p}}{(1-\theta)^{p}} + \frac{\sigma(\sigma+1)(\sigma+2)\phi^{p+2}}{(1-\phi)^{p+2}}\\ &\quad + \frac{(5+\zeta)\sigma(\sigma+1)\phi^{p+1}}{(1-\phi)^{p+1}} + \frac{(4-2\zeta)\sigma\phi^{p}}{(1-\phi)^{p}} \leq 2(1-\zeta)(1-\theta)^{\delta}\binom{p+\delta-2}{\delta-1}\theta^{p-1}\end{aligned} \tag{3.6}$$

holds. Then

$$P_{\theta,\phi}^{\delta,\sigma}(KH^0) \subset KH^0(\zeta).$$

Proof. It suffices to show that $P_{\theta,\phi}^{\delta,\sigma}(KH^0) \subset KH^0(\zeta)$. In essence, we must show that $\psi_4 \leq (1-\zeta)\theta^{p-1}$ where

$$\begin{aligned}\psi_4 &= \sum_{k=1}^{\infty}(k+p)(k+p-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}|a_{k+p}|\\ &\quad + \sum_{k=1}^{\infty}(k+p)(k+p+\zeta)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}|b_{k+p}|\end{aligned}$$

Since $f_p \in KH^0$, then

$$|a_{k+p}| \leq \frac{k+p+1}{2}, \quad and \quad |b_{k+p}| \leq \frac{k+p-1}{2}$$

$$\begin{aligned}
\psi_4 \leq \frac{1}{2}\Bigg[&\sum_{k=1}^{\infty}(k+p)(k+p+1)(k+p-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta} \\
&+\sum_{k=1}^{\infty}(k+p)(k+p-1)(k+p+\zeta)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}\Bigg] \\
=\frac{1}{2}\Bigg[&\sum_{k=1}^{\infty}(k+p-1)(k+p-2)(k+p-3)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta} \\
&+\sum_{k=1}^{\infty}(7-\zeta)(k+p-1)(k+p-2)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta} \\
&+\sum_{k=1}^{\infty}(10-4\zeta)(k+p-1)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta} \\
&+\sum_{k=1}^{\infty}2(1-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta} \\
&+\sum_{k=1}^{\infty}(k+p-1)(k+p-2)(k+p-3)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma} \\
&+\sum_{k=1}^{\infty}(5+\zeta)(k+p-1)(k+p-2)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}\Bigg] \\
&+\sum_{k=1}^{\infty}(4-2\zeta)(k+p-1)\binom{k+p+\sigma-2}{\sigma-1}\phi^{k+p-1}(1-\phi)^{\sigma}\Bigg]
\end{aligned}$$

$$\psi_4 \leq \frac{1}{2}\Bigg[\sum_{k=1}^{\infty}\delta(\delta+1)(\delta+2)\binom{k+p+\delta-2}{\delta+2}\theta^{k+p-1}(1-\theta)^{\delta}$$
$$+\sum_{k=1}^{\infty}(7-\zeta)\delta(\delta+1)\binom{k+p+\delta-2}{\delta+1}\theta^{k+p-1}(1-\theta)^{\delta}$$
$$+\sum_{k=1}^{\infty}(10-4\zeta)\delta\binom{k+p+\delta-2}{\delta}\theta^{k+p-1}(1-\theta)^{\delta}$$
$$+\sum_{k=1}^{\infty}2(1-\zeta)\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}(1-\theta)^{\delta}$$
$$+\sum_{k=1}^{\infty}\sigma(\sigma+1)(\sigma+2)\binom{k+p+\sigma-2}{\sigma+2}\phi^{k+p-1}(1-\phi)^{\sigma}$$
$$+\sum_{k=1}^{\infty}(5+\zeta)\sigma(\sigma+1)\binom{k+p+\sigma-2}{\sigma+1}\phi^{k+p-1}(1-\phi)^{\sigma}$$
$$+\sum_{k=1}^{\infty}(4-2\zeta)\sigma\binom{k+p+\sigma-2}{\sigma}\phi^{k+p-1}(1-\phi)^{\sigma}\Bigg]$$
$$=\frac{1}{2}\Bigg[\delta(\delta+1)(\delta+2)\theta^{3}(1-\theta)^{\delta}\sum_{k=0}^{\infty}\binom{k+p+\delta+1}{\delta+2}\theta^{k+p-1}$$
$$+(7-\zeta)\delta(\delta+1)\theta^{2}(1-\theta)^{\delta}\sum_{k=0}^{\infty}\binom{k+p+\delta}{\delta+1}\theta^{k+p-1}$$
$$+(10-4\zeta)\delta\theta(1-\theta)^{\delta}\sum_{k=0}^{\infty}\binom{k+p+\delta-1}{\delta}\theta^{k+p-1}$$
$$+2(1-\zeta)(1-\theta)^{\delta}\sum_{k=1}^{\infty}\binom{k+p+\delta-2}{\delta-1}\theta^{k+p-1}$$
$$+\sigma(\sigma+1)(\sigma+2)\phi^{3}(1-\phi)^{\sigma}\sum_{k=0}^{\infty}\binom{k+p+\sigma+1}{\sigma+2}\phi^{k+p-1}$$
$$+(5+\zeta)\sigma(\sigma+1)\phi^{2}(1-\phi)^{\sigma}\sum_{k=0}^{\infty}\binom{k+p+\sigma}{\sigma+1}\phi^{k+p-1}$$
$$+(4-2\zeta)\sigma\phi(1-\phi)^{\sigma}\sum_{k=0}^{\infty}\binom{k+p+\sigma-1}{\sigma}\phi^{k+p-1}\Bigg]$$

$$\begin{aligned}
\psi_4 \leq \frac{1}{2}\Bigg[&\delta(\delta+1)(\delta+2)\theta^3(1-\theta)^\delta \frac{\theta^{p-1}}{(1-\theta)^{p+\delta+2}} \\
&+ (7-\zeta)\delta(\delta+1)\theta^2(1-\theta)^\delta \frac{\theta^{p-1}}{(1-\theta)^{p+\delta+1}} \\
&+ (10-4\zeta)\delta\theta(1-\theta)^\delta \frac{\theta^{p-1}}{(1-\theta)^{p+\delta}} + 2(1-\zeta)(1-\theta)^\delta \frac{\theta^{p-1}}{(1-\theta)^{\delta}} \\
&+ \sigma(\sigma+1)(\sigma+2)\phi^3(1-\phi)^\sigma \frac{\phi^{p-1}}{(1-\phi)^{p+\sigma+2}} \\
&+ (5+\zeta)\sigma(\sigma+1)\phi^2(1-\phi)^\sigma \frac{\phi^{p-1}}{(1-\phi)^{p+\sigma+1}} \\
&+ (4-2\zeta)\sigma\phi(1-\phi)^\sigma \frac{\phi^{p-1}}{(1-\phi)^{p+\sigma}} - 2(1-\zeta)(1-\theta)^\delta \binom{p+\delta-2}{\delta-1}\theta^{p-1}\Bigg] \\
= \frac{1}{2}\Bigg[&\frac{\delta(\delta+1)(\delta+2)\theta^{p+2}}{(1-\theta)^{p+2}} + \frac{(7-\zeta)\delta(\delta+1)\theta^{p+1}}{(1-\theta)^{p+1}} + \frac{(10-4\zeta)\delta\theta^p}{(1-\theta)^p} \\
&+ 2(1-\zeta)\theta^{p-1} + \frac{\sigma(\sigma+1)(\sigma+2)\phi^{p+2}}{(1-\phi)^{p+2}} + \frac{(5+\zeta)\sigma(\sigma+1)\phi^{p+1}}{(1-\phi)^{p+1}} \\
&+ \frac{(4-2\zeta)\sigma\phi^p}{(1-\phi)^p} - 2(1-\zeta)(1-\theta)^\delta \binom{p+\delta-2}{\delta-1}\theta^{p-1}\Bigg]
\end{aligned}$$

Using 3.6,

$$\begin{aligned}
\psi_4 &\leq \frac{1}{2}[2(1-\zeta)(1-\theta)^\delta \binom{p+\delta-2}{\delta-1}\theta^{p-1} + 2(1-\zeta)\theta^{p-1} - 2(1-\zeta)(1-\theta)^\delta \binom{p+\delta-2}{\delta-1}\theta^{p-1}] \\
&= \frac{1}{2}[2(1-\zeta)\theta^{p-1}]
\end{aligned}$$

So, $\psi_4 \leq (1-\zeta)\theta^{p-1}$ and $P^{\delta,\sigma}_{\theta,\phi}(KH^0) \subset KH^0(\zeta)$.

4. Conclusion

In this study, several findings based on various subclasses of harmonic p-valent functions associated with Pascal Distribution Series are provided. Further, by utilizing the power series representation of the coefficients of the Pascal probability generating function, as well as the formation of the needed variable terms for this research, the aim of investigating the various subclasses of the harmonic p-valent functions through the Pascal generating function was achieved, with the results generated showcasing conformity with the given premises and lemmas, thereby remaining consistent to all assumptions made during the research.

References

[1] Y. Avici, E. Zlotkiewicz, *On harmonic univalent mappings*, Mariae Curie Sklodowska Sect., Ann. Univ. 44 (2009), 1-7.

[2] S. Çakmak, E. Yaşar, S. Yalçin, Ş. Altinkaya, *Some Connections Between Various Subclasses of Harmonic Univalent Functions Involving Pascal Distribution Series*, (2020).

[3] J. Clunie, T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. 9 (1984), 3-25.

[4] P. Duren, *Harmonic Mappings in the Plane*, Cambridge University Press (2004).

[5] J. Hamzat, R. El-Ashwah, *Some Properties of a Generalized Multiplier Transform on Analytic p-valent functions*, 74, 9 (2022), 1274-1283.

[6] J. Jahangiri, *Harmonic functions starlike in the unit disk*, J. Math. Anal. Appl. 235 (1999), 470-477.

[7] W. Nazeer, Q. Mehmood, S. Kang, A. Haq, *An application of Binomial distribution series on certain analytic functions*, J. Comp. Anal. App. 26 (2019), 11-17.

[8] A. Oladipo, *On subclasses of analytic and univalent functions*, Adv. Appl. Math. Anal. 4, 1 (2009), 87-93.

[9] S. Porwal, M. Kumar, *A unified study on starlike and convex functions associated with Poisson distribution series*, Afr. Math., 27 (2016), 10-21.

[10] H. Silverman, E. Silva, *Subclasses of harmonic univalent functions*, New Zeland J. Math. 28 (1999), 275-284.

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