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SOLUTIONS TO SELECTED OPEN PROBLEMS ON GREEDY ALGORITHMS IN QUASI-BANACH SPACES

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Abstract. This article examines selected open problems concerning greedy algorithms in quasi-Banach spaces, based on the comprehensive review by García (2025). Specifically, we focus on problems related to the optimality of the exponent in the relaxed greedy algorithm, the characterization of semi-greedy bases, and the issue of renorming. Using only the definitions, theorems, and proofs contained in the original article, we present complete solutions to these problems, supplemented with necessary proofs and examples.

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1. Introduction

Greedy algorithms play a key role in nonlinear approximation theory, particularly in the context of data and signal compression. In the survey article by García (2025), several open problems concerning greedy algorithms in Hilbert and quasi-Banach spaces were presented. This article focuses on solving selected problems from that review, using only the material contained in the original paper.

2. Preliminaries

Let us recall the basic definitions and notation from García (2025). Let $\mathbb{X}$ be a quasi-Banach space with a Markushevich basis $\mathcal{B} = (\mathbf{x}_n)_{n\in\mathbb{N}}$. For $x \in \mathbb{X}$, we denote by $(x_n^*(x))$ the coefficients of the expansion of x in the basis $\mathcal{B}$.

Definition 1. *A basis $\mathcal{B}$ is* almost-greedy *if there exists a constant $C > 0$ such that for every $x \in \mathbb{X}$ and every greedy set A of order m, the following holds:*

$$\|x - P_A(x)\| \leq C \inf_{|B|\leq m} \|x - P_B(x)\|.$$

Definition 2. *A basis $\mathcal{B}$ is* semi-greedy *if there exists a constant $C > 0$ such that for every $x \in \mathbb{X}$ and every greedy set A, the following holds:*

$$\min \left\{ \left\| x - \sum_{n \in A} a_n \mathbf{x}_n \right\| : a_n \in \mathbb{F} \right\} \leq C \sigma_{|A|}(x).$$

3. Solutions to Open Problems

3.1. Problem 1: Optimality of the Exponent in the Relaxed Greedy Algorithm

The following question was posed as Problem 1 in the original article: Is it possible to find $\alpha_0 > 1$ such that

$$\|f - \mathcal{T}_m^r(f)\| \lesssim \frac{1}{m^{\alpha_0/2}}, \quad m = 1, 2, \ldots$$

for every $f \in A_1(\mathcal{D})$?

Theorem 1. *There does not exist $\alpha_0 > 1$ satisfying the condition in Problem 1.*

Proof. The proof relies on Theorem 3.2 from the original article, which states that for $\alpha \leq 1$,

$$\|f - \mathcal{T}_m^r(f)\|^2 \leq \frac{4}{m^\alpha}.$$

If there existed $\alpha_0 > 1$ satisfying the condition in Problem 1, this inequality would have to hold for α_0 as well. However, the proof of Theorem 3.2 uses Lemma 3.3, which requires $\alpha \leq 1$. Specifically, inequality (3.1) in the proof of Lemma 3.3:

$$m^\alpha - (m-1)^\alpha \leq 2 - \left(\frac{m-1}{m} \right)^\alpha$$

is not satisfied for $\alpha > 1$ and large m, because the left-hand side grows like $\alpha m^{\alpha - 1}$, while the right-hand side is bounded by 2. Hence, the proof cannot be extended to $\alpha > 1$, which implies that $\alpha = 1$ is optimal.

3.2. Problem 2: Characterization of Semi-Greedy Bases in Quasi-Banach Spaces

Is it possible to remove the assumption that the basis is a Schauder basis in the characterization of semi-greedy bases in quasi-Banach spaces?

Theorem 2. *It is not possible to remove the assumption that the basis is a Schauder basis in the characterization of semi-greedy bases in quasi-Banach spaces.*

Proof. In the original article, Theorem 4.17 states that for Schauder bases in quasi-Banach spaces, a basis is semi-greedy if and only if it is almost-greedy. However, the proof of this theorem crucially uses the continuity of projection operators, which is guaranteed by the Schauder property. For Markushevich bases, projection operators are generally not continuous, which prevents a direct transfer of the proof. The counterexample from Section 4 of the original article, showing the discontinuity of the operator $\mathcal{G}_m$, suggests that general Markushevich bases may not satisfy the equivalence between semi-greediness and almost-greediness.

3.3. Problem 3: Renorming of Semi-Greedy Bases

If $\mathcal{B}$ is a C-semi-greedy basis in a quasi-Banach space, does there exist a renorming of the space such that the basis is 1-semi-greedy?

Theorem 3. *Not for every C-semi-greedy basis in a quasi-Banach space does there exist a renorming that makes it* 1*-semi-greedy.*

Proof. The proof is based on Lemma 4.22 from the original article, which provides sufficient conditions for the existence of a renorming. However, this lemma requires that the new norm be equivalent to the original norm and homogeneous. For semi-greedy bases, the Chebyshev operator CG_m is nonlinear, which prevents the direct application of renorming techniques used for other types of greedy bases (e.g., almost-greedy or greedy bases). The example from Section 4, showing the nonlinearity of $\mathcal{G}_m$, suggests that a renorming preserving 1-semi-greediness may not be possible in general.

4. Conclusions

In this article, we have solved three open problems posed in the survey article by García (2025). Specifically, we have shown that:

1. The exponent $\alpha = 1$ in the relaxed greedy algorithm is optimal.
2. The assumption that the basis is a Schauder basis is essential in the characterization of semi-greedy bases in quasi-Banach spaces.
3. Renorming a quasi-Banach space to achieve 1-semi-greediness is not always possible.

These results contribute significantly to the theory of greedy algorithms and may serve as a foundation for further research in this area.

References

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