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SOME RESULTS ON KENMOTSU MANIFOLDS ADMITTING SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. Four semi-symmetric classes are discussed in the present study and we show that a Kenmotsu manifold admitting semi-symmetric metric connection belonging the class C_3 is Einstein, whereas such a manifold belonging to each of the class C_1, C_2 and C_4 is η-Einstein. Also we consider Ricci solitons in Kenmotsu manifolds and prove condition for soliton vector field to be orthogonal to characteristic vector field.

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1. Introduction

In 2014 Shaikh and Kundu [17] established the equivalency of various geometric structures. They claimed the followings:

Class Name	Equivalent Classes
C_1	$R\cdot R=0, R\cdot P=0, R\cdot E=0, R\cdot P^*=0, R\cdot \mathcal{M}=0, R\cdot \mathcal{W}_i=0, R\cdot \mathcal{W}_i^*=0$
C_2	$E\cdot R=0, E\cdot P=0, E\cdot E=0, E\cdot P^*=0, E\cdot \mathcal{M}=0, E\cdot \mathcal{W}_i=0, E\cdot \mathcal{W}_i^*=0$
C_3	$R\cdot K=0, R\cdot C=0$
C_4	$E\cdot C=0, E\cdot K=0$
C_5	$R=0, E=0, P=0,\ P^*=0,\ \mathcal{M}=0, \mathcal{W}_i=0, \mathcal{W}_i^*=0$

where the symbols R, C, E, P, K, $\mathcal{M}$ and $\mathcal{W}_i$ stand for Riemman curvature tensor, conformal curvature tensor [7], concircular curvature tensor [19], projective curvature tensor [19], conharmonic curvature tensor [9], M-projective curvature tensor [13], $\mathcal{W}_i$-curvature tensor ([13], [14], [15]) and $\mathcal{W}_i^*$-curvature tensor, $i=1,2,...,9$.

Hayden [8] introduced the idea of semi-symmetric metric connections on a differential manifold and Yano [20] gave a systematic study of the semi-symmetric metric connection on a Riemannian manifold. Also semi-symmetric metric connection on contact manifolds have been studied by Baishya and Chowdhury ([1], [2]), Prasad et al. [16], Murthy et al. [11], Yadav et al. [18], Nagaraja and Premalatha [12] and many others. In this paper, we study Kenmotsu manifolds admitting semi-symmetric metric connection belonging to different semi-symmetric classes. Recently Bakshi et. al in ([3], [4]) studied various semi-symmetric classes on $(LCS)_n$-manifolds and α-cosymplectic manifolds respectively.

Definition 1. *A Kenmotsu manifold (M^n, g) is said to be η-Einstein manifold if its Ricci tensor S follows the condition*

$$S(X,Y) = xg(X,Y) + y\eta(X)\eta(Y)$$

for any smooth function x, y and vector fields X, Y on M^n. For $y = 0$ an η-Einstein manifold reduces to Einstein manifold.

The present paper is structured as follows. In section-2, we briefly recall some known results for Kenmotsu manifolds. Then, we study Kenmotsu manifolds admitting semi-symmetric metric connection belonging to the class C_i $(i = 1, 2, 3, 4, 5)$ and we prove that a Kenmotsu manifold belonging the class C_3 is Einstein, whereas such a manifold belonging to each of the class C_1, C_4 and C_4 is η-Einstein. In section 4, we consider Kenmotsu metric admitting semi symmetric connection as a Ricci soliton with soliton vector field as characteristic vector field.

2. Kenmotsu manifolds

Let M^n be a differential manifold of dimension $n = (2m+1)$. Now the structure (ϕ, ξ, η) where ϕ is a tensor field of type $(1,1)$, ξ a characteristic or Reeb vector field and η is an 1-form satisfying the following conditions

$$\phi^2 = -I + \eta \otimes \xi, \tag{1}$$

$$\eta(\xi) = 1, \tag{2}$$

$$\phi\xi = 0, \ \eta \circ \phi = 0, \ \text{rank}\phi = \frac{n-1}{2} \tag{3}$$

is said to be an **almost contact structure**. In general, a differentiable manifold M^n together with the almost contact structure (ϕ, ξ, η) is called an almost contact manifold (see Blair [6]) and it is denoted by (M^n, ϕ, ξ, η). Moreover, if g is a Riemannian metric on M^n satisfying

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{4}$$

$$g(X,\xi) = \eta(X), \tag{5}$$
$$g(\phi X, Y) = -g(X, \phi Y),$$

for any vector fields X, Y on M^n, then the manifold M^n is said to admit an almost contact metric structure $(\phi,\ \xi,\ \eta,\ g)$ and the manifold is called an almost contact metric manifold denoted by $(M^n,\ \phi,\ \xi,\ \eta,\ g)$. The fundamental 2-form Φ associate with the almost contact metric structure is defined by $\Phi(X,Y) = g(X, \phi Y)$. The structure is normal if the tensor field $N = [\phi,\phi] + 2d\eta \otimes \xi$ vanishes, where $[\phi,\phi]$ is the Nijenhuis torsion of ϕ. An almost contact metric manifold $(M^n,\ \phi,\ \xi,\ \eta,\ g)$ is said to be almost Kenmotsu manifold [10] if $d\eta = 0$, $d\Phi = 2\eta \wedge \Phi$. A normal almost Kenmotsu manifold is a Kenmotsu manifold and the normality condition is given by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X, Y on M^n. In a Kenmotsu manifold, for any vector fields X, Y, Z on M^n, the following relations hold

$$\nabla_X \xi = X - \eta(X)\xi, \tag{6}$$

$$(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y), \tag{7}$$

$$S(X,\xi) = -(n-1)\eta(X), \tag{8}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{9}$$

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{10}$$

$$(\nabla_Z R)(X,Y)\xi = g(X,Z)Y - g(Y,Z)X - R(X,Y)Z. \tag{11}$$

A linear connection ∇^s of (M^n, g) is said to be semi-symmetric metric connection if the the torsion tensor T^s of the connection ∇^s admits

$$T^s(X,Y) = \eta(Y)X - \eta(X)Y,$$

and

$$\nabla^s g = 0.$$

The relation between the semi-symmetric metric connection ∇^s and the Levi-Civita connection ∇ of (M^n, g) has been obtained by K.Yano [20], which is given by

$$\nabla^s_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi. \tag{12}$$

If R and R^s are the curvature tensors of the Levi-Civita connection ∇ and the semi-symmetric metric connection ∇^s respectively, then we have

$$\begin{aligned} R^s(X,Y)Z &= R(X,Y)Z + 3\{g(X,Z)Y - g(Y,Z)X\} \\ &\quad +2\eta(Z)\{\eta(Y)X - \eta(X)Y\} \\ &\quad +2\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi, \quad (13) \\ \eta(R^s(X,Y)Z) &= 2\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}, \quad (14) \\ S^s(Y,Z) &= S(Y,Z) - (3n-5)g(Y,Z) + 2(n-2)\eta(Y)\eta(Z), \quad (15) \\ S^s(Y,\xi) &= -2(n-1)\eta(Y) \quad (16) \\ r^s &= r - (3n^2 - 7n + 4), \quad (17) \end{aligned}$$

where S^s denotes the Ricci tensor w.r.t the semi symmetric metric connection.

The forms of Conformal [7], Conharmonic [9], Concircular [19] and Projective curvature tensor are

$$\begin{aligned} C(Z,V)X &= R(Z,V)X - \frac{1}{n-2}[S(V,X)Z - g(X,Z)QV + g(V,X)QZ - S(X,Z)V] \\ &\quad - \frac{r}{(n-2)(n-1)}[g(X,V)Z - g(X,Z)V], \quad (18) \\ K(Z,V)X &= R(Z,V)X - \frac{1}{n-2}[S(V,X)Z - g(X,Z)QV + g(V,X)QZ - S(X,Z)V] \quad (19) \\ E(Z,V)X &= R(Z,V)X - \frac{r}{n(n-1)}[g(X,V)Z - g(X,Z)V]. \quad (20) \\ P(X,Y,Z) &= R(X,Y,Z) - \frac{1}{n-1}[S(Y,Z)X - S(X,Z)Y]. \quad (21) \end{aligned}$$

In view of (14), from (18), (19) and (20) one can easily bring out the following:

$$\begin{aligned} &\eta(C^s(X,Y)Z)) \\ = &\left[\frac{r^s}{(n-1)(n-2)} - \frac{3(n-1)}{n-2}\right][g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] \\ + &\frac{1}{n-2}[S(X,Z)\eta(Y) - S(Y,Z)\eta(X)], \quad (22) \end{aligned}$$

$$\begin{aligned} &\eta(K^s(X,Y)Z)) \\ = &-\frac{3(n-1)}{n-2}[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] \\ &+\frac{1}{n-2}[S(X,Z)\eta(Y) - S(Y,Z)\eta(X)], \quad (23) \end{aligned}$$

$$\begin{aligned} &\eta(E^s(X,Y)Z)) \\ = &\left(\frac{r^s}{n(n-1)} + 2\right)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)]. \quad (24) \end{aligned}$$

3. Kenmotsu manifolds admitting semi-symmetric metric connection

In this section we consider different semi-symmetric classes on Kenmotsu manifold admitting a semi-symmetric metric connection and show that a Kenmotsu manifold belonging the class C_3 is Einstein, whereas such a manifold belonging to each of the class C_1, C_2 and C_4 is η-Einstein.

3.1. Kenmotsu manifold admitting a semi-symmetric metric connection belonging to the class C_1

We take a semi-symmetric structure from the semi-symmetric class C_1

$$(R^s(X,Y)\cdot R^s)(Z,U)V = 0.$$

Which yields

$$g(R^s(\xi,Y)R^s(Z,U)V,\xi) = g(R^s(R^s(\xi,Y)Z,U)V,\xi) \\ +g(R^s(Z,R^s(\xi,Y)U)V,\xi) + g(R^s(Z,U)R^s(\xi,Y)V,\xi).$$

Next taking an orthonormal frame field $\{e_1,e_2,e_3,...,e_{n-1},e_n=\xi\}$ of the manifold M^n and contrcting over Y and Z in the above equation, we get

$$\sum_{i=1}^{n} g(R^s(\xi,e_i)R^s(e_i,U)V,\xi) = \sum_{i=1}^{n} g(R^s(R^s(\xi,e_i)e_i,U)V,\xi) \\ +\sum_{i=1}^{n} g(R^s(e_i,R^s(\xi,e_i)U)V,\xi) + \sum_{i=1}^{n} g(R^s(e_i,U)R^s(\xi,e_i)V,\xi). \tag{25}$$

Using (7) to (11), we obtain

$$\sum_{i=1}^{n} g(R^s(\xi,e_i)R^s(e_i,U)V,\xi) \\ = 2(3n-7)g(U,V) - 4(n-3)\eta(U)\eta(V) - 2S(U,V),$$

$$\sum_{i=1}^{n} g(R^s(R^s(\xi,e_i)e_i,U)V,\xi) \\ = 4(n-1)[g(U,V) - \eta(U)\eta(V)],$$

$$\sum_{i=1}^{n} g(R^s(e_i,R^s(\xi,e_i)U)V,\xi) = 4[-g(U,V) + \eta(U)\eta(V)],$$

$$\sum_{i=1}^{n} g(R^s(e_i,U)R^s(\xi,e_i)V,\xi) = 4(n-1)\eta(U)\eta(V).$$

Now with the help of the above four equations, (25) yields

$$S(U,V) = (n-3)g(U,V) - 2(n-2)\eta(U)\eta(V)].$$

Thus, we state the following theorem;

Theorem 1. *A Kenmotsu manifold admitting a semi-symmetric metric connection belonging to the class C_1 is always an η-Einstein manifold.*

Observe that

$$\begin{aligned}
& (R(X,Y)\cdot R^s)(Z,U)V \\
= \ & (R(X,Y)\cdot R)(Z,U)V + 3\{g(Z,V)R(X,Y)U - g(U,V)R(X,Y)Z\} \\
& +2\eta(V)\{\eta(U)R(X,Y)Z - \eta(Z)R(X,Y)U\} \\
& +2R(X,Y)\xi\{g(U,V)\eta(Z) - g(Z,V)\eta(U)\}.
\end{aligned}$$

If we suppose $R\cdot R^s = 0$, then $R\cdot R = 0$ if and only if

$$\begin{aligned}
0 \ = \ & 3\{g(Z,V)R(X,Y)U - g(U,V)R(X,Y)Z\} \\
& +2\eta(V)\{\eta(U)R(X,Y)Z - \eta(Z)R(X,Y)U\} \\
& +2R(X,Y)\xi\{g(U,V)\eta(Z) - g(Z,V)\eta(U)\}.
\end{aligned}$$

From which we get

$$\begin{aligned}
R(X,Y)Z &= \eta(Z)\{\eta(X)Y - \eta(Y)X\}, \\
S(Y,Z) &= -(n-1)\eta(Y)\eta(Z).
\end{aligned}$$

Substituting these in (21), we get $P(X,Y)Z = 0$. Thus we can state the following

Theorem 2. *A Kenmotsu manifold M^n with $R\cdot R^s = 0$ possesses $R\cdot R = 0$ if and if M belongs to the class C_5.*

3.2. Kenmotsu manifold admitting a semi-symmetric metric connection belonging to the class C_2

We consider a semi-symmetric structure from the semi-symmetric class C_2

$$(E^s(X,Y)\cdot R^s)(Z,U)V = 0,$$

Which gives

$$g(E^s(\xi,Y)R^s(Z,U)V,\xi) = g(R^s(E^s(\xi,Y)Z,U)V,\xi)$$
$$+g(R^s(Z,E^s(\xi,Y)U)V,\xi) + g(R^s(Z,U)E^s(\xi,Y)V,\xi).$$

Next taking an orthonormal frame field of the manifold M^n and contrcting over Y and Z in the above equation, we get

$$\sum_{i=1}^{n} g(E^s(\xi,e_i)R^s(e_i,U)V,\xi) = \sum_{i=1}^{n} g(R^s(E^s(\xi,e_i)e_i,U)V,\xi)$$
$$+\sum_{i=1}^{n} g(R^s(e_i,E^s(\xi,e_i)U)V,\xi) + \sum_{i=1}^{n} g(R^s(e_i,U)E^s(\xi,e_i)V,\xi). \quad (26)$$

Using (7) to (11) and (24) we obtain

$$\sum_{i=1}^{n} g(E^s(\xi,e_i)R^s(e_i,U)V,\xi)$$
$$= [\frac{r^s}{n(n-1)} + 2][(3n-7)g(U,V) - 2(n-3)\eta(U)\eta(V) - S(U,V)],$$

$$\sum_{i=1}^{n} g(R^s(E^s(\xi,e_i)e_i,U)V,\xi)$$
$$= 2(n-1)[\frac{r^s}{n(n-1)} + 2][g(U,V) - \eta(U)\eta(V)],$$

$$\sum_{i=1}^{n} g(R^s(e_i,E^s(\xi,e_i)U)V,\xi)$$
$$= -2[\frac{r^s}{n(n-1)} + 2][g(U,V) - \eta(U)\eta(V)],$$

$$\sum_{i=1}^{n} g(R^s(e_i,U)E^s(\xi,e_i)V,\xi)$$
$$= 2(n-1)[\frac{r^s}{n(n-1)} + 2]\eta(U)\eta(V).$$

Now in mind of the above four equations, (26) yields

$$\begin{aligned} \left[2+\frac{r^s}{n(n-1)}\right][-S(U,V) \\ +(n-3)g(U,V)-2(n-2)\eta(U)\eta(V)] &= 0. \end{aligned}$$

Thus, we state the following theorem;

Theorem 3. *A Kenmotsu manifold admitting a semi-symmetric metric connection belonging to the semi-symmetric class C_2 is either an η-Einstein manifold or the manifold is of constant scalar curvature with respect to semi-symmetric metric connection.*

3.3. Kenmotsu manifold admitting a semi-symmetric metric connection belonging to the class C_3

We consider a semi-symmetric structure from the semi-symmetric class C_3

$$(R^s(X,Y)\cdot K^s)(Z,U)V=0,$$

which implies

$$\begin{aligned} &g(R^s(\xi,Y)K^s(Z,U)V,\xi)=g(K^s(R^s(\xi,Y)Z,U)V,\xi) \\ &+g(K^s(Z,R^s(\xi,Y)U)V,\xi)+g(K^s(Z,U)R^s(\xi,Y)V,\xi). \end{aligned}$$

Next taking an orthonormal frame field of the manifold M^n and contrcting over Y and Z in the above equation, we get

$$\begin{aligned} &\sum_{i=1}^{n} g(R^s(\xi,e_i)K^s(e_i,U)V,\xi)=\sum_{i=1}^{n} g(K^s(R^s(\xi,e_i)e_i,U)V,\xi) \\ &+\sum_{i=1}^{n} g(K^s(e_i,R^s(\xi,e_i)U)V,\xi)+\sum_{i=1}^{n} g(K^s(e_i,U)R^s(\xi,e_i)V,\xi). \end{aligned} \tag{27}$$

Using (7)-(11) and (23) we have

$$\begin{aligned} &\sum_{i=1}^{n} g(R^s(\xi,e_i)K^s(e_i,U)V,\xi) \\ = &-\frac{6(n-1)}{n-2}[\eta(U)\eta(V)-g(U,V)]+\frac{2r^s}{(n-2)}g(U,V) \\ &-\frac{2}{n-2}[S(U,V)+(n-1)\eta(U)\eta(V)], \end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{n} g(K^s(R^s(\xi,e_i)e_i,U)V,\xi)\\ =\ &\frac{6(n-1)^2}{n-2}[\eta(U)\eta(V)-g(U,V)]\\ &+\frac{2(n-1)}{n-2}[S(U,V)+(n-1)\eta(U)\eta(V)],\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{n} g(K^s(e_i,R^s(\xi,e_i)U)V,\xi)\\ =\ &-\frac{6(n-1)}{n-2}[\eta(U)\eta(V)-g(U,V)]\\ &-\frac{2}{n-2}[S(U,V)+(n-1)\eta(U)\eta(V)],\end{aligned}$$

$$\sum_{i=1}^{n} g(K^s(e_i,U)R^s(\xi,e_i)V,\xi)=-\frac{8(n-1)^2}{n-2}\eta(U)\eta(V).$$

Now in mind of the above four equations, (27) yields

$$S(U,V)=\left[\frac{r^s+3(n-1)^2}{(n-1)}\right]g(U,V).$$

Thus, we state the following theorem;

Theorem 4. *A Kenmotsu manifold admitting a semi-symmetric metric connection belonging to the semi-symmetric class C_3 is always an Einstein manifold.*

3.4. Kenmotsu manifolds admitting a semi-symmetric metric connection belonging to the class C_4

We consider a semi-symmetric structure from the semi-symmetric class C_4

$$(E^s(X,Y)\cdot K^s)(Z,U)V=0,$$

which implies

$$\begin{aligned}&g(E^s(\xi,Y)K^s(Z,U)V,\xi)=g(K^s(E^s(\xi,Y)Z,U)V,\xi)\\ &+g(K^s(Z,E^s(\xi,Y)U)V,\xi)+g(K^s(Z,U)E^s(\xi,Y)V,\xi).\end{aligned}$$

Next taking an orthonormal frame field of the manifold M^n and contrcting over Y and Z in the above equation, we get

$$\sum_{i=1}^{n} g(E^s(\xi, e_i)K^s(e_i, U)V, \xi) = \sum_{i=1}^{n} g(K^s(E^s(\xi, e_i)e_i, U)V, \xi)$$
$$+\sum_{i=1}^{n} g(K^s(e_i, E^s(\xi, e_i)U)V, \xi) + \sum_{i=1}^{n} g(K^s(e_i, U)E^s(\xi, e_i)V, \xi). \qquad (28)$$

Using (7)-(11) and (23), (24) we have

$$\begin{aligned}&\sum_{i=1}^{n} g(E^s(\xi, e_i)K^s(e_i, U)V, \xi)\\ = \;& \{\frac{r^s}{n(n-1)} + 2\}[-\frac{3(n-1)}{n-2}\{\eta(U)\eta(V) - g(U,V)\} + \frac{r^s}{(n-2)}g(U,V)\\ & -\frac{1}{n-2}\{S(U,V) + (n-1)\eta(U)\eta(V)\}],\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{n} g(K^s(E^s(\xi, e_i)e_i, U)V, \xi)\\ = \;& \frac{3(n-1)^2}{n-2}\{\frac{r^s}{n(n-1)} + 2\}[\eta(U)\eta(V) - g(U,V)]\\ & +\frac{(n-1)}{n-2}\{\frac{r^s}{n(n-1)} + 2\}[S(U,V) + (n-1)\eta(U)\eta(V)],\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{n} g(K^s(e_i, E^s(\xi, e_i)U)V, \xi)\\ = \;& -\frac{3(n-1)}{n-2}\{\frac{r^s}{n(n-1)} + 2\}[\eta(U)\eta(V) - g(U,V)]\\ & -\frac{1}{n-2}\{\frac{r^s}{n(n-1)} + 2\}[S(U,V) + (n-1)\eta(U)\eta(V)],\end{aligned}$$

$$\begin{aligned}&\sum_{i=1}^{n} g(K^s(e_i, U)E^s(\xi, e_i)V, \xi)\\ = \;& \left[-\frac{3(n-1)^2}{n-2} + (r+n-1)\right]\{\frac{r^s}{n(n-1)} + 2\}\eta(U)\eta(V).\end{aligned}$$

Now taking the above four equations in mind (28) yields

$$\begin{aligned} 0 &= [\frac{r^s}{n(n-1)} + 2][S(U,V) - \{3(n-1) + \frac{r^s}{(n-1)}\}g(U,V) \\ &\quad + \{2n-3 + \frac{r(n-2)}{(n-1)}\}\eta(U)\eta(V)]. \end{aligned}$$

Thus, we state the following theorem;

Theorem 5. *A Kenmotsu manifold admitting a semi-symmetric metric connection belonging to the class C_4 is either an η-Einstein manifold or the manifold is of constant scalar curvature with respect to semi-symmetric metric connection.*

4. Kenmotsu metric admitting semi-symmetric metric connection as a Ricci solitons

A Ricci soliton is a natural generalization of an Einstein metric and is defined on a Riemannian manifold(M,g) by:

$$(L_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{29}$$

where V is a vector field on M, g and S denote the metric tensor and its Ricci tensor respectively, L_V denotes the Lie-derivative operator along V , and λ is a real constant. The Ricci soliton is said to be shrinking, steady, and expanding according as λ is negative, zero, and positive respectively. A more general notion of this is a η-Ricci soliton defined on an almost contact metric manifold (M,ϕ,ξ,η,g) by [5]

$$(L_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y), \tag{30}$$

where μ is a real constant.
Let (M,ϕ,ξ,η,g) be a Kenmotsu manifold admitting semi symmetric metric connection ∇^s. Suppose $(M,\, g,\, \lambda^s)$ is a Ricci soliton with respect to the semi symmetric metric connection ∇^s with ξ as soliton vector field. Then from (12), we have

$$\nabla^s_X\xi = \nabla_X\xi + X - \eta(X)\xi,$$

and

$$\begin{aligned} &(L^s_\xi g)(X,Y) + 2S^s(X,Y) + 2\lambda^s g(X,Y) \\ = \;& g(\nabla^s_X\xi, Y) + g(X, \nabla^s_Y\xi) + 2\{S(X,Y) - (3n-5)g(X,Y) \\ & + 2(n-2)\eta(X)\eta(Y)\} + 2\lambda^s g(X,Y), \\ = \;& (L_\xi g)(X,Y) + 2S(X,Y) + 2\{\lambda^s - 3n + 6\}g(X,Y) + 2(2n-5)\eta(X)\eta(Y). \end{aligned}$$

Thus we state the following:

Theorem 6. *If (M, g, λ^s) is a Ricci soliton with respect to the semi symmetric metric connection ∇^s, then (g, ξ, λ, μ) with $\lambda = \{\lambda^s - 3n + 6\}$, $\mu = (2n-5)$ is a η-Ricci soliton with respect to Levi-Civita connection ∇.*

Suppose (M, g, V, λ) and (M, g, V, λ^s) are Ricci solitons with respect to ∇ and ∇^s respectively. Then from (30), we have

$$\begin{aligned} & (L_V{}^s g)(X,Y) + 2S^s(X,Y) + 2\lambda^s g(X,Y) \\ = \; & (L_V g)(X,Y) + 2S(X,Y) + 2\lambda^s g(X,Y) \\ & -\{g(X,V)\eta(Y) + g(Y,V)\eta(X)\} + 2g(X,Y)\eta(V). \end{aligned}$$

If $\lambda = \lambda^s + \eta(V)$, then

$$g(X,V)\eta(Y) + g(Y,V)\eta(X) = 0,$$

which implies $\eta(V) = 0$ or $V \perp \xi$, therefore $\lambda = \lambda^s$. This leads to the following:

Theorem 7. *In a Kenmotsu manifold, if g is a Ricci soliton with respect to both ∇ and ∇^s with soliton constants λ and λ^s, then $\lambda = \lambda^s$, and $V \perp \xi$.*

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