

## GENERALIZATION COMPLEX NUMBERS AND COMPLEX FUNCTION

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**ABSTRACT.** In this paper, the author using the generalization complex numbers. For these new numbers is defined complex function, the existence conditions of derivation, complex functions.

The presented findings are the results of the author's original research.

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### 1. GENERALIZATION COMPLEX NUMBERS

In the paper [1] the author define generalization complex numbers. In this paper I use the following result:

**Definition 1.** *The generalization complex numbers  $w_1, w_2 \in C_{G_x}$ ,  $w_1 = x_1 + y_1i + t_1j$ ,  $w_2 = x_2 + y_2i + t_2j$ ,  $(x_k, y_k, t_k) \in \mathbb{R}^3$ ,  $k = \overline{1, 2}$ ,  $i = j = \sqrt{-1}$ , are equal,  $w_1 = w_2$  if :*

$$\begin{cases} x_1 &= x_2 \\ y_1 &= y_2 \\ t_1 &= t_2 \end{cases}$$

Generalization complex numbers  $w_1, w_2 \in C_{G_x}$ ,  $w_1 = a \cdot i$ ,  $w_2 = b \cdot j$ ,  $a, b \in \mathbb{R}^*$ ,  $i = j = \sqrt{-1}$ , are equal  $w_1 = w_2$ , if  $a = b$ .

### 2. COMPLEX FUNCTION

Let be  $C_{G_x} = \{w, w = x + yi + tj, (x, y, t) \in \mathbb{R}^3, i^2 = j^2 = -1\}$ ,  $E \subseteq C_{G_x}$ ,  $C_{G_x}$  the set of generalization complex numbers.

**Definition 2.** *The application  $f : E \rightarrow G_{G_x}$ ,  $W = f(w)$ ,  $E \subseteq C_{G_x}$  to name complex function with the generalization complex variable  $w \in E$ .*

This function is considered that the function with the generalization complex variable  $w = x + yi + tj$ ,  $w \in E$  or that the complex function with independent variables  $x, y, t$ ,  $(x, y, t) \in R^3$ .

We note

$$f(w) = f(x + yi + tj) = U(x, y, t) + iV(x, y, t) + jT(x, y, t).$$

We have

$$\begin{aligned} \operatorname{Re} f(w) &= U(x, y, t), \\ \operatorname{Im}_1 f(w) &= V(x, y, t), \\ \operatorname{Im}_2 f(w) &= T(x, y, t). \end{aligned}$$

We observe that the definition of the function  $f(w)$  is equivalent with the definition of three real functions with the real variables  $x, y, t$ . Geometrical, the definition of function  $f(w)$  we interpret by the definition of punctual transformation ( $L$ ) of lots  $E$  from the domain complex variable  $w$ , in the lots  $f(L)$  from the complex domain  $E'$  of variable  $W = f(w)$ .

**Definition 3.** *The function  $f : E \rightarrow C_{G_x}$ ,  $E \subseteq C_{G_x}$ ,  $u = f(w)$  is injectiv (univalent), if for all  $w_1, w_2 \in E$ ,  $w_1 \neq w_2$ , we have  $f(w_1) \neq f(w_2)$ .*

**Definition 4.** *The function  $f : E \rightarrow E'$ ,  $E, E' \subseteq C_{G_x}$ ,  $u = f(w)$  is surjectiv if  $E' = f(E)$*

**Definition 5.** *The function  $f : E \rightarrow E'$ ,  $u = f(w)$ , injectiv and surjectiv is bijectiv.*

**Definition 6.** *The function  $f : E \rightarrow C_{G_x}$ ,  $E \subseteq C_{G_x}$ ,  $u = f(w)$  has the limit  $\ell \in C_{G_x}$  when  $w$  tend to  $w_0 \in E$ ,  $w_0 = x_0 + y_0i + t_0j$ , the notation  $\lim_{w \rightarrow w_0} f(w) = \ell$ , if for every  $\epsilon > 0$ , exists the number  $\delta(\epsilon)$  and for  $|w - w_0| < \delta$  it results that  $|f(w) - \ell| < \epsilon$ .*

**Definition 7.** *The function  $f : E \rightarrow C_{G_x}$ ,  $W = f(w)$ ,  $E \subseteq C_{G_x}$  is continue in  $w_0 \in E$ ,  $w_0 = x_0 + y_0i + t_0j$ , if*

$$\lim_{w \rightarrow w_0} f(w) = f(w_0). \quad (1)$$

**Remark 2.7** *The continue function in every point from  $E$  is the function continue in  $E$ .*

Let be the complex functions  $f : E \rightarrow C_{G_x}$ ,  $g : E \rightarrow C_{G_x}$ ,  $u = f(w)$ ,  $s = g(w)$  and  $w_0 \in E$ . We have the properties

$$\lim_{w \rightarrow w_0} [f(w) + g(w)] = \lim_{w \rightarrow w_0} f(w) + \lim_{w \rightarrow w_0} g(w), \quad (2)$$

$$\lim_{w \rightarrow w_0} [f(w) \times g(w)] = \lim_{w \rightarrow w_0} f(w) \times \lim_{w \rightarrow w_0} g(w), \quad (3)$$

$$\lim_{w \rightarrow w_0} \frac{f(w)}{g(w)} = \frac{\lim_{w \rightarrow w_0} f(w)}{\lim_{w \rightarrow w_0} g(w)}, \quad g(w) \neq 0 \text{ in } E. \quad (4)$$

Analogous we define the complex functions, if we consider the lots  $C_{G_y}$  and  $C_{G_t}$ . Analogous we define the limit of functions, continue functions  $f : E_1 \rightarrow C_{G_y}$ ,  $E_1 \subseteq C_{G_y}$ ,  $w_0 \in E_1$ ,  $g : E_2 \rightarrow C_{G_t}$ ,  $E_2 \subseteq C_{G_t}$ ,  $w_0 \in E_2$ .

## 2.1. DERIVATION OF THE COMPLEX FUNCTION

**Definition 2.1.1** *The complex function  $f : E \rightarrow C_{G_x}$ ,  $W = f(w)$ ,  $E \subseteq C_{G_x}$  is derivative or monogene in  $w_0 \in E$ ,  $w_0 = x_0 + y_0i + t_0j$ , if the function  $g(w) = \frac{f(w) - f(w_0)}{w - w_0}$ ,  $g : E \setminus \{w_0\} \rightarrow C_{G_x}$ , has the limit in  $w_0 \in E$ , again the limit is the derivation the complex function  $f$  in  $w_0$ , with the notation  $f'(w_0)$  and*

$$f'(w_0) = \lim_{w \rightarrow w_0} \frac{f(w) - f(w_0)}{w - w_0}. \quad (5)$$

**Definition 2.1.2** *The complex function  $f : E \rightarrow C_{G_x}$ ,  $W = f(w)$ ,  $E \subseteq C_{G_x}$ , is derivative or olomorpe in the lots  $D \subseteq E$ , if the function  $f$  is derivative in every point of  $D$ .*

If  $f$  is derivative in  $D$ , then we have new complex function  $f' : D \rightarrow C_{G_x}$ ,  $W = f'(w)$ , which in every point  $w \in D$ , has the value  $f'(w)$ . For the complex functions  $f : E \rightarrow C_{G_x}$ ,  $g : E \rightarrow C_{G_x}$ ,  $E \subseteq C_{G_x}$ ,  $u = f(w)$ ,  $s = g(w)$  we have the next derivative rules:

$$[f(w) + g(w)]' = f'(w) + g'(w), \quad (6)$$

$$[f(w) \times g(w)]' = f'(w) \times g(w) + f(w) \times g'(w) \quad (7)$$

$$\left[ \frac{f(w)}{g(w)} \right]' = \frac{f'(w) \times g(w) - f(w) \times g'(w)}{g^2(w)}, \quad g(w) \neq 0 \text{ in } E, \quad (8)$$

$$[(f \circ g)(w)]' = f'(g(w)) \times g'(w), \quad (9)$$

$$[f^{-1}(w)]' = \frac{1}{f'(w)}, \quad \text{with } f'(w) \neq 0, \quad (10)$$

where  $f^{-1} : f(E) \rightarrow E$ ,  $v = f^{-1}(w)$  is to invert of function  $f$ , with the condition that  $f$  is bijectiv.

We have the existence conditions of derivation in the point  $w_0 \in E$  for the complex function  $f : E \rightarrow C_{G_x}$ ,  $W = f(w)$ ,  $E \subseteq C_{G_x}$  in the next theorems.

**Theorem 2.1.3.** For that the complex function

$$f : E \rightarrow G_{G_x}, E \subseteq C_{G_x}, W = f(w), w = x + yi + tj,$$

$$f(w) = f(x + yi + tj) = U(x, y, t) + iV(x, y, t) + jT(x, y, t), \quad (11)$$

to be derivate (monogene) in the point  $w_0 \in E$ ,  $w_0 = x_0 + y_0i + t_0j$  is necessary that the functions  $U$ ,  $V$  and  $T$  to be partial derivative by first order in the point  $(x_0, y_0, t_0) \in R^3$  and satisfies the derivative (monogene) conditions :

$$\left\{ \begin{array}{l} \frac{\partial U}{\partial x}(x_0, y_0, t_0) = \frac{\partial V}{\partial y}(x_0, y_0, t_0), \\ \frac{\partial U}{\partial y}(x_0, y_0, t_0) = -\frac{\partial V}{\partial x}(x_0, y_0, t_0), \\ \frac{\partial U}{\partial x}(x_0, y_0, t_0) = \frac{\partial T}{\partial t}(x_0, y_0, t_0), \\ \frac{\partial U}{\partial t}(x_0, y_0, t_0) = -\frac{\partial T}{\partial x}(x_0, y_0, t_0), \\ \frac{\partial U}{\partial y}(x_0, y_0, t_0) = \frac{\partial U}{\partial t}(x_0, y_0, t_0). \end{array} \right. \quad (12)$$

**Proof.** For  $w \in E \setminus \{w_0\}$ ,  $w = x + yi + tj$  we have

$$\begin{aligned} & \frac{f(w) - f(w_0)}{w - w_0} = \quad (13) \\ & = \frac{U(x, y, t) - U(x_0, y_0, t_0) + i[V(x, y, t) - V(x_0, y_0, t_0)] + j[T(x, y, t) - T(x_0, y_0, t_0)]}{x - x_0 + i(y - y_0) + j(t - t_0)}. \end{aligned}$$

We suppose that  $w \rightarrow w_0$  on parallel way with real axis  $Ox$ , then

$$\left\{ \begin{array}{l} y = y_0 \\ \text{and } x \rightarrow x_0. \\ t = t_0 \end{array} \right. \quad (14)$$

Using (2.14), from (2.13) we obtain

$$\begin{aligned} f'(w_0) &= \lim_{x \rightarrow x_0} \frac{U(x, y_0, t_0) - U(x_0, y_0, t_0)}{x - x_0} + \quad (15) \\ &+ i \lim_{x \rightarrow x_0} \frac{V(x, y_0, t_0) - V(x_0, y_0, t_0)}{x - x_0} + \\ &+ j \lim_{x \rightarrow x_0} \frac{T(x, y_0, t_0) - T(x_0, y_0, t_0)}{x - x_0}, \end{aligned}$$

and hence, we have

$$f'(w_0) = \frac{\partial U}{\partial x}(x_0, y_0, t_0) + i \frac{\partial V}{\partial x}(x_0, y_0, t_0) + j \frac{\partial T}{\partial x}(x_0, y_0, t_0). \quad (16)$$

Let be  $w \rightarrow w_0$  on parallel way with real axis  $Oy$ , then

$$\begin{cases} x = x_o \\ \text{and } y \rightarrow y_0. \\ t = t_o \end{cases} \quad (17)$$

From (2.13) we obtain

$$f'(w_0) = \frac{1}{i} \frac{\partial U}{\partial y}(x_0, y_0, t_0) + \frac{\partial V}{\partial y}(x_0, y_0, t_0) + \frac{j}{i} \frac{\partial T}{\partial y}(x_0, y_0, t_0). \quad (18)$$

Analogous, for  $w \rightarrow w_0$  on parallel way with real axis  $Ot$ , then

$$\begin{cases} x = x_o \\ \text{and } t \rightarrow t_0 \\ y = y_o \end{cases} \quad (19)$$

and from (2.13) we have

$$f'(w_0) = \frac{1}{j} \frac{\partial U}{\partial t}(x_0, y_0, t_0) + \frac{i}{j} \frac{\partial V}{\partial t}(x_0, y_0, t_0) + \frac{\partial T}{\partial x}(x_0, y_0, t_0). \quad (20)$$

From (2.16) and (2.18) we obtain

$$\begin{aligned} \frac{\partial U}{\partial x}(x_0, y_0, t_0) + i \frac{\partial V}{\partial x}(x_0, y_0, t_0) + j \frac{\partial T}{\partial x}(x_0, y_0, t_0) = \\ \frac{1}{i} \frac{\partial U}{\partial y}(x_0, y_0, t_0) + \frac{\partial V}{\partial y}(x_0, y_0, t_0) + \frac{j}{i} \frac{\partial T}{\partial y}(x_0, y_0, t_0). \end{aligned} \quad (21)$$

By multiplication with  $i$  the relation (2.21), since  $i \times i = i \cdot i \cos 0^\circ = -1$ ,  $i \times j = i \cdot j \cos 90^\circ = 0$ ,  $i \times \frac{1}{i} = i \times \frac{1 \times (-i)}{i \times (-i)} = i \times (-i) = 1$ ,  $i \times \frac{j}{i} = i \times \frac{j \times (-i)}{i \times (-i)} = 0$ , we obtain

$$i \frac{\partial U}{\partial x}(x_0, y_0, t_0) - \frac{\partial V}{\partial x}(x_0, y_0, t_0) = \frac{\partial U}{\partial y}(x_0, y_0, t_0) + i \frac{\partial V}{\partial y}(x_0, y_0, t_0) \quad (22)$$

and hence, we have

$$\begin{cases} \frac{\partial U}{\partial x}(x_0, y_0, t_0) = \frac{\partial V}{\partial y}(x_0, y_0, t_0), \\ \frac{\partial U}{\partial y}(x_0, y_0, t_0) = -\frac{\partial V}{\partial x}(x_0, y_0, t_0). \end{cases} \quad (23)$$

Using (2.16) and (2.20) we have

$$\begin{aligned} \frac{\partial U}{\partial x}(x_0, y_0, t_0) + i \frac{\partial V}{\partial x}(x_0, y_0, t_0) + j \frac{\partial T}{\partial x}(x_0, y_0, t_0) &= \\ = \frac{1}{j} \frac{\partial U}{\partial t}(x_0, y_0, t_0) + \frac{i}{j} \frac{\partial V}{\partial t}(x_0, y_0, t_0) + \frac{\partial T}{\partial t}(x_0, y_0, t_0). \end{aligned} \quad (24)$$

By multiplication with  $j$  the relation (2.24), we obtain

$$j \frac{\partial U}{\partial x}(x_0, y_0, t_0) - \frac{\partial T}{\partial x}(x_0, y_0, t_0) = \frac{\partial U}{\partial t}(x_0, y_0, t_0) + j \frac{\partial T}{\partial t}(x_0, y_0, t_0) \quad (25)$$

and hence, we have

$$\begin{cases} \frac{\partial U}{\partial x}(x_0, y_0, t_0) = \frac{\partial T}{\partial t}(x_0, y_0, t_0), \\ \frac{\partial U}{\partial t}(x_0, y_0, t_0) = -\frac{\partial T}{\partial x}(x_0, y_0, t_0). \end{cases} \quad (26)$$

From relations (2.21) and (2.24) it results that

$$\begin{aligned} \frac{1}{i} \frac{\partial U}{\partial y}(x_0, y_0, t_0) + \frac{\partial V}{\partial y}(x_0, y_0, t_0) + \frac{j}{i} \frac{\partial T}{\partial y}(x_0, y_0, t_0) &= \\ = \frac{1}{j} \frac{\partial U}{\partial t}(x_0, y_0, t_0) + \frac{i}{j} \frac{\partial V}{\partial t}(x_0, y_0, t_0) + \frac{\partial T}{\partial t}(x_0, y_0, t_0). \end{aligned} \quad (27)$$

Using (2.23) and (2.26) we have

$$\frac{\partial V}{\partial y} = \frac{\partial T}{\partial t},$$

again

$$\frac{1}{i} = \frac{1 \times (-i)}{i \times (-i)} = -i, \quad \frac{i}{j} = \frac{i \times (-j)}{j \times (-j)} = 0, \quad \frac{1}{j} = \frac{1 \times (-j)}{j \times (-j)} = -j.$$

From (2.27) we obtain

$$i \frac{\partial U}{\partial y}(x_0, y_0, t_0) = j \frac{\partial U}{\partial t}(x_0, y_0, t_0). \quad (28)$$

Using Definition 1.1, the relations (2.22) and (2.28) we obtain

$$\frac{\partial U}{\partial y}(x_0, y_0, t_0) = \frac{\partial U}{\partial t}(x_0, y_0, t_0). \quad (29)$$

From (2.23), (2.26) and (2.29) we obtain the relations (2.12).

**Remark 2.1.4** *Let be  $T(x, y, 0) = 0$ , then from (2.11) we obtain*

$$f(w) = f(z) = U(x, y, 0) + iV(x, y, 0) \quad (30)$$

$$f : C \rightarrow C, w = z = x + iy \in C, w_0 = z_0 = x_0 + iy_0 \in C.$$

In this case, from Theorem 2.1.3 we obtain the derivative conditions of the function  $f$  with the variable  $z$ :

$$\begin{cases} \frac{\partial U}{\partial x}(x_0, y_0, 0) = \frac{\partial V}{\partial y}(x_0, y_0, 0), \\ \frac{\partial U}{\partial y}(x_0, y_0, 0) = -\frac{\partial V}{\partial x}(x_0, y_0, 0) \end{cases} \quad (31)$$

which are the derivative conditions Cauchy - Riemann for the function  $f$  defined by (2.30).

**Remark 2.1.5** *Let be  $V(x, 0, t) = 0$ , then from (2.11) we obtain*

$$f(w) = f(z) = U(x, 0, t) + jT(x, 0, t) \quad (32)$$

$f : E_1 \rightarrow C_1, C_1 = \{z \mid z = x + tj, j^2 = -1\}, w_0 = z_0 = x_0 + t_0j, E_1 \subseteq C_1, f(z)$  the image of  $z$  by  $f$ ,  $Ox$  real axis.

In this case, from Theorem 2.1.3 we obtain the derivative conditions of the function  $f$  with the complex variable  $z$ :

$$\begin{cases} \frac{\partial U}{\partial x}(x_0, 0, t_0) = \frac{\partial T}{\partial t}(x_0, 0, t_0), \\ \frac{\partial U}{\partial t}(x_0, 0, t_0) = -\frac{\partial T}{\partial x}(x_0, 0, t_0) \end{cases} \quad (33)$$

which are the derivative conditions on type Cauchy - Riemann.

**Remark 2.1.6** *Let be  $U(0, y, t) = 0$ , then from (2.11) we obtain*

$$f(w) = iV(0, y, t) + jT(0, y, t) \quad (34)$$

$f : E \rightarrow C_{G_x}$ ,  $w = iy + tj$ ,  $w_0 = iy_0 + jt_0 \in E$ ,  $E \subseteq C_{G_x}$ ,  $O(y)$ ,  $O(t)$  imaginary axis.

In this case, from Theorem 2.1.3 we obtain the derivative condition of the function  $f$ :

$$\frac{\partial V}{\partial y}(0, y_0, t_0) = \frac{\partial T}{\partial t}(0, y_0, t_0). \quad (35)$$

**Theorem 2.1.7** *Let be the complex function  $g : E \rightarrow C_{G_y}$ ,  $E \subseteq C_{G_y}$ ,  $W = g(w)$ ,  $w = y + tj + xk$ ,*

$$g(w) = U(y, t, x) + jV(y, t, x) + kT(y, t, x) \quad (36)$$

*The function  $g$  is derivative (monogene) in the point  $w_0 \in E$ ,  $w_0 = y_0 + t_0j + x_0k$ , if the functions  $U$ ,  $V$  and  $T$  are the functions with partial derivations by first order in the point  $(y_0, t_0, x_0) \in R^3$  and satisfies the derivative conditions:*

$$\left\{ \begin{array}{l} \frac{\partial U}{\partial y}(y_0, t_0, x_0) = \frac{\partial V}{\partial t}(y_0, t_0, x_0), \\ \frac{\partial U}{\partial t}(y_0, t_0, x_0) = -\frac{\partial V}{\partial y}(y_0, t_0, x_0), \\ \frac{\partial U}{\partial y}(y_0, t_0, x_0) = \frac{\partial T}{\partial x}(y_0, t_0, x_0), \\ \frac{\partial U}{\partial x}(y_0, t_0, x_0) = -\frac{\partial T}{\partial y}(y_0, t_0, x_0), \\ \frac{\partial U}{\partial t}(y_0, t_0, x_0) = \frac{\partial U}{\partial x}(y_0, t_0, x_0). \end{array} \right. \quad (37)$$

**Proof.** Analogous of Theorem 2.1.3 proof, we obtain the monogene conditions (2.37) for the function  $g$  defined by (2.36).

**Remark 2.1.8** *Let be  $T(y, t, 0) = 0$ , then from (2.36) we obtain*

$$g(w) = g(z) = U(y, t, 0) + jV(y, t, 0) \quad (38)$$

$$g : E_1 \rightarrow C_1, \quad C_1 = \{z \mid z = y + tj, j^2 = -1\}, \quad E_1 \subseteq C_1 \subseteq C_{G_y}, \\ w_0 = z_0 = y_0 + t_0j \in E_1, \quad O(y) \quad \text{real axis.}$$

In this case, from Theorem 2.1.7 we obtain the derivative conditions of the function  $g$  with the complex variable  $z$ ,  $z \in C_1$ :

$$\left\{ \begin{array}{l} \frac{\partial U}{\partial y}(y_0, t_0, 0) = \frac{\partial V}{\partial t}(y_0, t_0, 0), \\ \frac{\partial U}{\partial t}(y_0, t_0, 0) = -\frac{\partial V}{\partial y}(y_0, t_0, 0), \end{array} \right. \quad (39)$$

which are the derivative conditions of type Cauchy - Riemann.

**Remark 2.1.9** Let be  $V(y, o, x) = 0$ , then from (2.36) we obtain

$$h(w) = U(y, x, 0) + kT(y, x, 0), \quad (40)$$

$h : E_2 \rightarrow C_2$ ,  $C_2 = \{z \mid z = y + xk, k^2 = -1\}$ ,  $E_2 \subseteq C_2$ ,  $C_2 \subseteq C_{G_y}$ ,  $Oy$  real axis.

From Theorem 2.1.7 we obtain the monogene (derivative) conditions of the function  $h$  whith the complex variable  $z$ ,  $z \in C_2$ :

$$\begin{cases} \frac{\partial U}{\partial y}(y_0, x_0, 0) = \frac{\partial T}{\partial x}(y_0, x_0, 0), \\ \frac{\partial U}{\partial x}(y_0, x_0, 0) = -\frac{\partial T}{\partial y}(y_0, x_0, 0), \end{cases} \quad (41)$$

the derivative conditions of type Cauchy - Riemann.

**Remark 2.1.10** Let be  $U(0, t, x) = 0$ , then from (2.36) we obtain

$$\ell(w) = jV(0, t, x) + kT(0, t, x) \quad (42)$$

$\ell : E_3 \rightarrow C_3$ ,  $C_3 = \{w \mid w = tj + xk, j^2 = k^2 = -1\}$ ,  $C_3 \subseteq C_{G_y}$ ,  $E_3 \subseteq C_3$ .

The derivative condition of function  $\ell$  is

$$\frac{\partial V}{\partial y}(0, t_0, x_0) = \frac{\partial T}{\partial x}(0, t_0, x_0). \quad (43)$$

**Theorem 2.1.11** Let be the complex function  $p : E \rightarrow C_{G_t}$ ,  $E \subseteq C_{G_t}$ ,  $W = p(w)$ ,  $w = t + xk + yi$ ,

$$p(w) = U(t, x, y) + kV(t, x, y) + iT(t, x, y). \quad (44)$$

The function  $p$  is derivative in the point  $w_0 \in E$ ,  $w_0 = t_0 + x_0k + y_0i$ , if the functions  $U$ ,  $V$  and  $T$  are partial derivative of first order in the point  $(t_0, x_0, y_0)$  and satisfases the derivative conditions:

$$\begin{cases} \frac{\partial U}{\partial y}(t_0, x_0, y_0) = \frac{\partial V}{\partial x}(t_0, x_0, y_0), \\ \frac{\partial U}{\partial x}(t_0, x_0, y_0) = -\frac{\partial V}{\partial t}(t_0, x_0, y_0), \\ \frac{\partial U}{\partial t}(t_0, x_0, y_0) = \frac{\partial T}{\partial y}(t_0, x_0, y_0), \\ \frac{\partial U}{\partial y}(t_0, x_0, y_0) = -\frac{\partial T}{\partial t}(t_0, x_0, y_0), \\ \frac{\partial U}{\partial x}(t_0, x_0, y_0) = \frac{\partial U}{\partial y}(t_0, x_0, y_0). \end{cases} \quad (45)$$

## 2.2. SINGULAR POINTS OF COMPLEX FUNCTION

**Definition 2.2.1.** Let be the complex function  $f : E \rightarrow C_{G_x}$ ,  $E \subseteq C_{G_x}$ . The point  $a \in E$  is the ordinary point for the function  $f$ , if exist the neighbourhood sphere of the point  $a$ , where  $f$  is derivative.

**Definition 2.2.2.** The point  $w_0 \in E$  is zero of order  $n$ ,  $n \in N$ , if the function  $f$  is to form  $f(w) = (w - w_0)^n \times g(w)$ ,  $g(w_0) \neq 0$  and the function  $g$  is derivative in the neighbourhood sphere of the point  $w_0$ .

**Definition 2.2.3.** The singular points of the function  $f : E \rightarrow C_{G_x}$   $E \subseteq C_{G_x}$  are all the points of  $C_{G_x}$ , which not are ordinary points of the function  $f$ .

**Definition 2.2.4.** The singular point  $w_0 \in C_{G_x}$  is pole for the complex function  $f$ , of the order  $k$ ,  $k \in N$ , if is zero for the function  $\frac{1}{f(w)}$ .

**Definition 2.2.5.** Any singular point of the complex function  $f$  which not is pole, is essential singular point.

**Definition 2.2.6.** The apparent singular point is the point  $w_0 \in E \subseteq C_{G_x}$ , for which the function  $f$  not is defined, therefore exist the finite limit  $\alpha = \lim_{w \rightarrow w_0} f(w)$ .

**Definition 2.2.7.** The complex function  $f$  is integer function if  $f$  is derivative in the lots  $C_{G_x}$ .

**Definition 2.2.8.** The complex function  $f : E \rightarrow C_{G_x}$  is meromorphe function in  $E$ ,  $E \subseteq C_{G_x}$ , if the function  $f$  has in  $E$  only the singular points which are poles.

**Remark 2.2.9.** Analogous we define these notions for the lots  $C_{G_y}$  and  $C_{G_t}$ .

## 2.3. ELEMENTARY COMPLEX FUNCTIONS

### 2.3.1. POLYNOMYAL FUNCTION

$$f(w) = P(w) = \sum_{p=0}^n a_p w^p, \quad a_p \in C_{G_x}, \quad p = \overline{0, n}, \quad w \in C_{G_x}. \quad (46)$$

The point  $\infty$  is pole of order  $n$  for the polinomyal function  $f(w)$ .

### 2.3.2. RATIONAL FUNCTION

$$g(w) = \frac{P(w)}{Q(w)} = \frac{\sum_{k=0}^n a_k w^k}{\sum_{k=0}^n b_k w^k}, \quad (47)$$

where  $P, Q$  are polynoms.

### 2.3.3. EXPONENTIAL FUNCTION

Exponential function is

$$f : C_{G_x} \rightarrow C_{G_x}, f(w) = e^w, w \in C_{G_x} \quad (48)$$

$$w = x + iy + jt.$$

From (1.48) we obtain

$$f(w) = e^w = e^{x+yi+jt} = e^x \cdot e^{iy} e^{jt},$$

and hence

$$f(w) = r e^{i\varphi} e^{j\theta}, \quad (49)$$

where  $r = |f(w)| = e^x$ ,  $\varphi = \arg_1 f(w) = y$ ,  $\theta = \arg_2 f(w) = t$ .

### 2.3.4. LOGARITHMICALLY FUNCTION

The logarithmically function  $\ln w$  is defined that inverse exponential function. By  $\ln w$  we note the lots of functions  $W_p(w)$ ,  $p \in Z$ , which verifies the equation  $e^W = w$ . Let be

$$w = \rho e^{i(\eta+2p\pi)} e^{j(\mu+2p\pi)}$$

and  $W = A + Bi + Cj$ . We have the equation

$$e^{A+Bi+Cj} = \rho e^{i(\eta+2p\pi)} e^{j(\mu+2p\pi)}$$

and hence,  $e^A = \rho$ ,  $B = \eta + 2p\pi$ ,  $C = \mu + 2p\pi$ , then

$$W_p(w) = \ln w = \ln \rho + i(\eta + 2p\pi) + j(\mu + 2p\pi), \quad (50)$$

$$W_p : C_{G_x} \setminus \{0\} \rightarrow C_{G_x}.$$

### 2.3.5. POWER COMPLEX FUNCTION

Power complex function is  $w^\alpha$  ( $\alpha = constant$ ) and is defined using logarithmically function, by  $w^\alpha = e^{\alpha \ln w}$ ,  $\forall w \in C_{G_x} \setminus \{0\}$ .

### 2.3.6. RADICAL FUNCTION

$W = f(w) = \sqrt[n]{w}$ ,  $w \in C_{G_x}$  is defined that inverse of function  $w = W^n$ .

## 2.4. SPECIAL CLASSES COMPLEX FUNCTIONS

**Definition 2.4.1.** Let be the set generalization complex numbers  $C_{G_x}$ . The open sphere is the set

$$U_p = \{w \in C_{G_x} : |w| < 1\}. \quad (51)$$

**Definition 2.4.2.** Let be the set generalization complex numbers  $C_{G_x}$ . Outside open sphere  $U_p$  is the set

$$U_p^- = \{w \in C_{G_x} : |w| > 1\}. \quad (52)$$

**Definition 2.4.3.** The class derivative and univalent functions in  $U_p$  is the class functions  $T$  define by  $f(w) = w + a_2w^2 + a_3w^3 + \dots$ ,  $a_i \in C_{G_x}$ ,  $i \in N^* \setminus \{1\}$ ,  $w \in C_{G_x}$ ,  $|w| < 1$ , with the conditions  $f(0) = 0$ ,  $f'(0) = 1$ .

We note  $H(U_p)$  the set derivative functions in the open sphere  $U_p$  and we note

$$L_n = \{f \in H(U_p) : f(w) = w + a_{n+1}w^{n+1} + \dots\}, \quad T = L_1, \quad n \in N \setminus \{0\}. \quad (53)$$

**Definition 2.4.4.** The class meromorphe functions  $V$  with only pole  $\eta = \infty$  and univalent in  $U_p^-$  is defined by  $g(\eta) = \eta + b_0 + \frac{b_1}{\eta} + \dots$ ,  $\eta \in U_p^-$ ,  $b_i \in C_{G_x}$ ,  $i \in N$ .

We note

$$V_0 = \{g \in V : g(\eta) \neq 0, \eta \in U_p^-\}.$$

**Definition 2.4.5.** The function  $h$  derivative in  $U_p$  is in the class  $\mathcal{P}_p$ , if  $h(0) = 1$  and  $\operatorname{Re} h(w) > 0$ , for all  $w \in U_p$ .

**Definition 2.4.6.** The function  $f \in T$ , is real type if

$$\operatorname{Re} \left[ \frac{1-w^2}{w} \times f(w) \right] > 0, \quad \forall w \in U_p.$$

**Definition 2.4.7.** The function  $f$ , derivative in  $U_p$  is starlike if  $f(0) = 0$ ,  $f'(0) \neq 0$  and

$$\operatorname{Re} \frac{w \times f'(w)}{f(w)} > 0, \quad \forall w \in U_p.$$

**Definition 2.4.8.** The function  $f$ , derivative in  $U_p$ , is convex if  $f'(0) \neq 0$  and

$$\operatorname{Re} \frac{w \times f''(w)}{f'(w)} + 1 > 0, \quad \forall w \in U_p.$$

**Definition 2.4.9.** The function  $f$ , derivative in  $U_p$  with  $f(0) = 0$ ,  $f'(0) = 1$  is neighbour convex if exist the function  $g \in C_{G_x}$  and

$$\operatorname{Re} \frac{f'(w)}{g'(w)} > 0, \quad \forall w \in U_p.$$

**Definition 2.4.10.** *Let be the function  $f$  derivative in  $U_p$ ,  $f(0) = 0$ ,  $f'(0) = 1$  and  $\alpha$  real number. The function  $f$  is  $\alpha$ -convex if*

$$\operatorname{Re} \left[ (1 - \alpha) \frac{w \times f'(w)}{f(w)} + \alpha \left( \frac{w \times f''(w)}{f'(w)} + 1 \right) \right] > 0, \quad \forall w \in U_p.$$

**Definition 2.4.11.** *The function  $f$  derivative in  $U_p$ ,  $f(0) = 0$ ,  $f'(0) = 1$  is  $\alpha$ -starlike,  $\alpha \in C_{G_x}$ , if the function*

$$F_\alpha(w) = (1 - \alpha) \times f(w) + (\alpha \times w) \times f'(w)$$

*is starlike and*

$$\frac{f(w) \times f'(w)}{w} \neq 0, \quad \forall w \in U_p.$$

#### REFERENCES

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