Acta Universitatis Apulensis No. 78/2024
ISSN: 1582-5329 pp. 47-54
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2024.78.03

SOME INEQUALITIES INVOLVING THE DISTANCE IN METRIC SPACES

M.O Bello, A. A. Mogbademu

Abstract. In this paper, we provide a lower bound for the sum

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\rho_i\rho_j d^m(u_i,u_j) \tag{1.1}$$

where (X,d) is a metric space, $\{u_i\}_{i=1}^N$, $m,N\in N$ with $\rho_i\geq 0$ and $\sum_{i=1}^N\rho_i=1$ Some applications in normed linear spaces are also given.

2010 *Mathematics Subject Classification*: 26D15, 54E40 and 54E50

Keywords: Metric space, Inequality, Binomial expansion, upper bound, lower bound.

1. Introduction and Preliminaries

Inequalities have been a subject of great interest in mathematical analysis, it plays a crucial role in establishing relationships between mathematical expressions and its integral to various theorems and proofs. In this section, we recall some basic definitions and results in literature, see [6].

Definition 1. *A metric space is defined as a pair (X,d), where X is a set and $d: X\times X\to\mathbb{R}^+$ is a metric. This metric d satisfies the following properties for any elements u,v,w in X:*

- *(d1): $d(u,v)\geq 0$;*
- *(d2): $d(u,v)=0$ if and only if $u=v$;*
- *(d3): $d(u,v)=d(v,u)$ (symmetry);*
- *(d4): $d(u,w)\leq d(u,v)+d(v,w)$ (Triangle inequality).*

Definition 2. *Let X be a linear space over a field $\mathbb{K}$. A function $\|\cdot,\cdot\|$ defined by $\|\cdot,\cdot\| : X \to [0,\infty]$ is called a normed space if the following conditions are satisfied for all $u, v, w \in X$:*

- *(N1)* $\|x\| \geq 0$;
- *(N2)*$\|x\| = 0$ *if and only if* $x = 0$;
- *(N3)* $\|\lambda x\| = |\lambda|\|x\|, \quad \lambda \in \mathbb{K}$;
- *(N4)* $\|x + y\| \leq \|x\| + \|y\|$.

Dragomir and Gosa [1] established the inequality stated below in metric spaces.

Theorem 1. *Let (X, d) be a metric space and $u_i \in X,\ \rho_i \geq 0,\ i \in \{1, ..., N\}$ with $\sum_{i=1}^{N} \rho_i = 1$. Then we have the inequality*

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_i \rho_j d(u_i, u_j) \leq \inf_{u \in X} \left[\sum_{i=1}^{N} \rho_i d(u_i, u) \right]. \tag{1.2}$$

The inequality is sharp in the sense that the multiplicative coefficient (say $\lambda = 1$) in front of 'inf' cannot be replaced by a smaller number. Dragomir [4] further obtained the lower bound for the sum in (1.1) which is stated as follows:

Theorem 2. *Let (X, d) be a metric space and $m, N \in \mathbb{N}, N \geq 2$, $u_i \in X, \rho_i \geq 0, i \in \{1, ..., N\}$ with $\sum_{i=1}^{N} \rho_i = 1$. Then we have the inequality*

$$\left(\frac{2}{1 - \sum_{i=1}^{N} \rho_i^2} \right)^{m-1} \left(\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_i \rho_j d(u_i, u_j) \right)^{m} \leq \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_i \rho_j d(u_i, u_j)^m. \tag{1.3}$$

See [4] for proof. In [2] Hassan and Bessem obtained new inequalities in metric spaces. Indeed, they proved the following for the upper bound of the sum $\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_i \rho_j d^m(u_i, u_j)$.

Theorem 3. *Let (X, d) be a metric space and $m, N \in \mathbb{N}, N \geq 2$, $u_i \in X$, $\rho_i \geq 0, i \in \{1, ..., N\}$ with $\sum_{i=1}^{N} \rho_i = 1$. Then we have the inequality*

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_i \rho_j d(u_i, u_j)^m$$

$$\leq \frac{1}{2} \inf_{u \in X} \left[2 \sum_{i=1}^{N} \rho_i d(u_i, u)^m + \sum_{k=1}^{m-1} \binom{m}{k} \left(\sum_{i=1}^{N} \rho_i d^k(u, u_i) \right) \left(\sum_{i=1}^{N} \rho_i d^{m-k}(u_i, u) \right) \right]. \tag{1.4}$$

See [2] for proof and geometric interpretations. Motivated by the result of Hassan and Bessem [2], it is natural to ask:
If the lower bound of the sum in (1.1) *can be provided?*
It is the purpose of this paper to give an affirmation to the natural question using an elementary technique. Some applications of our results are provided.

2. Main Results

Theorem 4. *Let* (X, d) *be a metric space and* $m, N \in \mathbb{N}, N \geq 2$, $u_i \in X$, $\rho_i \geq 0$, $i \in \{1, ..., N\}$ *with* $\sum_{i=1}^{N} \rho_i = 1$. *Then we have the inequality as follows*

$$\frac{1}{2}\left[2\sum_{i=1}^{N}\rho_i d(u_i,u)^m + \sum_{k=1}^{m-1}(-1)^k\binom{m}{k}\left(\sum_{i=1}^{N}\rho_i d^k(u,u_i)\right)\left(\sum_{i=1}^{N}\rho_i d^{m-k}(u_i,u)\right)\right]$$

$$\leq \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\rho_i\rho_j d(u_i,u_j)^m \tag{2.1}$$

for all $u \in X$.

Proof. By triangle inequality, it follows that;

$$d(u_i,u) - d(u,u_j) \leq d(u_i,u_j).$$

Therefore,

$$[d(u_i,u) - d(u,u_j)]^m \leq d^m(u_i,u_j). \tag{2.2}$$

By expanding the LHS of (2.2) using binomial expansion and taking the power $m \geq 1$ in (2.2) we have

$$d^m(u_i,u_j) \geq [d(u_i,u) - d(u,u_j)]^m = \sum_{k=0}^{m}(-1)^k\binom{m}{k}d^{m-k}(u_i,u)d^k(u,u_j) \qquad (*)$$

where

$$\binom{m}{k} = \frac{m!}{(m-k)!k!}.$$

Re-writing (*) to have;

$$d^m(u_i,u_j) \geq \sum_{k=0}^{m}(-1)^k\binom{m}{k}d^{m-k}(u_i,u)d^k(u,u_j). \tag{2.3}$$

Multiplying (2.3) by $\rho_i\rho_j$ and summing over i and j from 1 to N, we get

$$\sum_{1\le i,j\le N}\rho_i\rho_j d^m(u_i,u_j) \ge \sum_{1\le i,j\le N}\rho_i\rho_j\sum_{k=0}^{m}(-1)^k\binom{m}{k}d^{m-k}(u_i,u)d^k(u,u_j)$$
$$=\sum_{k=0}^{m}(-1)^k\binom{m}{k}\sum_{1\le i,j\le N}\rho_i\rho_j d^{m-k}(u_i,u)d^k(u,u_j)$$
$$=\sum_{k=0}^{m}(-1)^k\binom{m}{k}\sum_{i=1}^{N}\rho_i d^{m-k}(u_i,u)\sum_{j=1}^{N}\rho_j d^k(u,u_j). \qquad (2.4)$$

We now have that;

$$\sum_{1\le i,j\le N}\rho_i\rho_j d^m(u_i,u_j)$$
$$\ge \sum_{i=1}^{N}\rho_i d^m(u_i,u)\sum_{1=1}^{N}\rho_i + \left[\sum_{k=1}^{m-1}(-1)^k\binom{m}{k}\left(\sum_{i=1}^{N}\rho_i d^{m-k}(u_i,u)\right)\left(\sum_{i=1}^{N}\rho_i d^{m-k}(u,u_i)\right)\right]$$
$$=2\sum_{i=1}^{N}\rho_i d^m(u_i,u) + \left[\sum_{k=1}^{m-1}(-1)^k\binom{m}{k}\left(\sum_{i=1}^{N}\rho_i d^{m-k}(u_i,u)\right)\left(\sum_{i=1}^{N}\rho_i d^{m-k}(u,u_i)\right)\right],$$

It is easy to see that

$$\sum_{1\le i,j\le N}\rho_i\rho_j d^m(u_i,u_j) = 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\rho_i\rho_j d(u_i,u_j)^m \qquad (2.5)$$

So, (2.5) becomes

$$2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\rho_i\rho_j d(u_i,u_j)^m$$
$$\ge 2\sum_{i=1}^{N}\rho_i d^m(u_i,u) + \left[\sum_{k=1}^{m-1}(-1)^k\binom{m}{k}\left(\sum_{i=1}^{N}\rho_i d^{m-k}(u_i,u)\right)\left(\sum_{i=1}^{N}\rho_i d^{m-k}(u,u_i)\right)\right].$$

That is,

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\rho_i\rho_j d(u_i,u_j)^m$$
$$\ge \frac{1}{2}\left[2\sum_{i=1}^{N}\rho_i d(u_i,u)^m + \sum_{k=1}^{m-1}(-1)^k\binom{m}{k}\left(\sum_{i=1}^{N}\rho_i d^k(u,u_i)\right)\left(\sum_{i=1}^{N}\rho_i d^{m-k}(u_i,u)\right)\right].$$

Hence the result in (2.1) is proved for all $u \in X$.

Remark 1. *Since $u \in X$ we observe that*

$$\sum_{1\le i,j\le N} \rho_i\rho_j d(u_i,u)d(u_j,u) = \sum_{k=1}^{N} \rho_k d(u_k,u)^2$$

i.e.,

$$\left(\sum_{i=1}^{N} \rho_i d^k(u,u_i)\right)\left(\sum_{i=1}^{N} \rho_i d^{m-k}(u_i,u)\right) = \left(\sum_{k=1}^{N} \rho_k d(u_k,u)\right)^{2m} \tag{2.6}$$

Combining (2.4) and (2.6) to have

$$\sum_{1\le i,j\le N} \rho_i\rho_j d^m(u_i,u_j) \ge \sum_{k=0}^{m}(-1)^k\binom{m}{k}\left(\sum_{r=1}^{N}\rho_r d(u_r,u)\right)^{2m}.$$

This implies,

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \rho_i\rho_j d(u_i,u_j)^m \ge \frac{1}{2}\left[\sum_{k=0}^{m}(-1)^k\binom{m}{k}\left(\sum_{r=1}^{N}\rho_r d(u_r,u)\right)^{2m}\right]$$

$$= \frac{1}{2}\left[\left(\sum_{r=1}^{N}\rho_r d(u_r,u)\right)^{2m} + \sum_{k=1}^{m-1}(-1)^k\binom{m}{k}\left(\sum_{r=1}^{N}\rho_r d(u_r,u)\right)^{2m}\right].$$

Thus,

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} \rho_i\rho_j d(u_i,u_j)^m$$

$$\ge \frac{1}{2}\left[\left(\sum_{r=1}^{N}\rho_r d(u_r,u)\right)^{2m} + \sum_{k=1}^{m-1}(-1)^k\binom{m}{k}\left(\sum_{r=1}^{N}\rho_r d(u_r,u)\right)^{2m}\right]$$

for $i,j,r \in \{1,...,N\}$.

Corollary 5. *Let (X,d) be a metric space, let $m,N \in \mathbb{N}, N \ge 2$ and $u_i \in X, i \in \{1,...,N\}$. Then we have the following inequality*

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N} d(u_i,u_j)^m$$

$$\ge N\sum_{i=1}^{N} d(u_i,u)^m + \frac{1}{2}\left[\sum_{k=1}^{m-1}(-1)^k\binom{m}{k}\left(\sum_{i=1}^{N} d^k(u,u_i)\right)\left(\sum_{i=1}^{N} d^{m-k}(u_i,u)\right)\right]. \tag{2.7}$$

Using (2.1) with $\rho_i = \frac{1}{N}, i \in \{1, ..., N\}$, we deduce that,

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \frac{1}{N^2} d(u_i, u_j)^m$$

$$\geq \sum_{i=1}^{N} \frac{1}{N} d(u_i, u)^m + \frac{1}{2} \left[\sum_{k=1}^{m-1} (-1)^k \binom{m}{k} \left(\sum_{i=1}^{N} \frac{1}{N} d^k(u, u_i) \right) . \sum_{i=1}^{N} \frac{1}{N} d^{m-k}(u_i, u) \right].$$

Multiplying through by N^2, we have

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} d(u_i, u_j)^m$$

$$\geq N \sum_{i=1}^{N} d(u_i, u)^m + \frac{1}{2} \left[\sum_{k=1}^{m-1} (-1)^k \binom{m}{k} \left(\sum_{i=1}^{N} d^k(u, u_i) \right) \left(\sum_{i=1}^{N} d^{m-k}(u_i, u) \right) \right].$$

for $u_i, u \in X, i \in \{1, ..., N\}$ and the inequality (2.7) is hence proved.

Corollary 6. *Let (X, d) be a metric space, let $m, N \in \mathbb{N}, N \geq 2$ and $u_i \in X, i \in \{1, ..., N\}$. Then,*

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} d^2(u_i, u_j) \geq N \sum_{i=1}^{N} d^2(u_i, u) - \left(\sum_{i=1}^{N} d(u_i, u) \right)^2. \tag{2.8}$$

Proof. Taking $m = 2$ in (2.8) one can deduce that

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} d(u_i, u_j)^2 \geq N \sum_{i=1}^{N} d^2(u_i, u) - \frac{1}{2} \left[1(2). \sum_{i=1}^{N} d(u, u_i). \sum_{i=1}^{N} d(u_i, u) \right]$$

$$= N \sum_{i=1}^{N} d^2(u_i, u) - \left(\sum_{i=1}^{N} d(u_i, u) \right)^2$$

for all $u \in X$ and the inequality (2.7) is proved.

3. Applications

Theorem 7. *Given that $(X, \|.\|)$ is a normed linear space and $u \in X$, $m \in \mathbb{N}$, $N \geq 2$, $\rho_i \geq 0$, $i, j, r = \{1, ..., N\}$ with $\sum_{i=1}^{N} \rho_i = 1$, then by (2.6) we have*

$$\sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho_i \rho_j \|u_i, u_j\|^m$$

$$\geq \frac{1}{2}\left[\sum_{i=1}^{N}\rho_i\|u_i,u\|^{2m}+\sum_{k=1}^{m-1}(-1)^k\binom{m}{k}\left(\sum_{r=1}^{N}\rho_r\|u_r,u\|\right)^{2m}\right] \tag{2.9}$$

for all $u \in X$*.*

Theorem 8. *Let* $(X,\|.\|)$ *be a normed linear space, let* $m, N \in \mathbb{N}, N \geq 2$ *with* $\{u_i\}_{i=1}^N \in X$*, then we have*

$$\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\|u_i,u_j\|^2 \geq N\sum_{i=1}^{N}\|u_i,u\| - \left(\sum_{i=1}^{N}\|u_i,u\|\right)^2 \tag{3.0}$$

for all $u \in X$*.*

References

[1] S.S Dragomir, A. C Gosa, *An inequality in metric spaces*, Journal of The Indonesian Mathematical Society, vol. 11,no.1, pp. 33-38,2005.

[2] H. Aydi, B. Samet, *On some metric inequalities and applications*, Journal of Function Space. Vol.2020, 3842879.

[3] M.O Bello, A.A Mogbademu, *Some inequalities in metric -type spaces*, Journal of the Nigerian Mathematical Society, vol.45, pp. 110-119,2026.

[4] S.S Dragomir, *Some power inequalities for the distance in metric spaces*, Preprint RGMIA Res. Rep. Coll. 23 (2020), Art. 115, 6 pp.

[5] S.S Dragomir, *Refined inequalities for the distance in metric spaces*, Preprint RGMIA Res. Rep. Coll. 23 (2020), Art. 119, 7 pp.

[6] J.R Giles, *Introduction to the Analysis of Metric Spaces*, Cambridge University Press, Cambridge, 1987.

Muinat O. Bello
Department of Mathematics, Faculty of Science,
University of Lagos, Akoka,
Lagos, Nigeria.
E-mail: *muinatbello11@gmail.com*

Adesanmi A. Mogbademu
Department of Mathematics, Faculty of Science,
University of Lagos, Akoka,
Lagos, Nigeria.
E-mail: *amogbademu@unilag.edu.ng*