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ON THE SEQUENTIAL HENSTOCK STIELTJES INTEGRABLE FUNCTIONS IN N-NORMED SPACES

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Abstract. In this paper, we define the Sequential Henstock Stieltjes integrable functions in n-normed space setting. Using this definition, we give some properties of the Sequential Henstock Stieltjes Integral taking values an n-normed space. We employ this to reformate the contractive definition in the context of n-normed spaces.

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1. Introduction

The concept of Henstock integral, established to remedy the deficiencies of the Riemann integral was introduced independently in the mid-1950s by R. Henstock and J. Kurzweil respectively. It is a useful generalisation of the Riemann integral and powerful to handle nowhere-continuous functions, extreme oscillatory functions (see [1-14]). While the standard definition of the Henstock integral uses the Riemann sums and $\varepsilon - \delta$ definition, the Sequential Henstock integral which involves the use of sequence of guage functions was introduced. Paxton[12] proved a theorem of a specific definition for Topological Henstock integral which was refined and called the Sequential Topological Henstock integral over a compact subspace. In the last one decade, several studies for varieties of generalized Riemann-type integrals for certain classes of functions have been considered by many researchers in order to improve on the approach of integration, see for example,[1 and 10] and the references therein.

We denote $\mathbb{R}$ and $\mathbb{N}$ as set of real and natural numbers respectively and $\ll$ as much more smaller.

A gauge on $[a, b]$ is a positive real-valued function $\delta : [a, b] \to \mathbb{R}^+$. This gauge is δ-fine if $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$.

A sequence of tagged partition P_n of [a,b] is a finite collection of ordered pairs $P_n = \{(u_{(i-1)_n}\, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ where $[u_{i-1}, u_i] \in [a, b]$, $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ and $a = u_0 < u_{i_1} < ... < u_{m_n} = b$.

2. Basic Definitions

We recall the following definitions (see [5-12]).

Definition 1. *([7,11]) Let X be a real vector space of dimension d, where $n \leq d$. A mapping $\|.,...,.\| : X^n \to \mathbb{R}$ which satisfies the following four conditions:*
(nN_1) $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,
(nN_2) $\|x_1, x_2, ..., x_n\|$ is invariant under any permutation,
(nN_3) $\|ax_1, x_2, ..., x_n\| = |a|\|x_1, x_2, ..., x_n\|$ for every $a \in \mathbb{R}$,
(nN_4) $\|x + x_1, x_2, ..., x_n\| \leq \|x, x_2, ..., x_n\| + \|x_1, x_2, ..., x_n\|$ is called n-norm on X. The pair $(X, \|.,...,.\|)$ for every $x_1, x_2, ..., x_n \in X$ is called n-normed spaces.

Example 1. *([13]) Let $X = \mathbb{R}^n$ with:*

$$\|x_1, ..., x_n\|_E = abs\left(\begin{bmatrix} x_{1_1} & ... & x_{1_n} \\ . & . & . \\ . & . & . \\ . & . & . \\ x_{n_1} & ... & x_{n_n} \end{bmatrix}\right)$$

where $x_i = (x_{i_1}, ..., x_{i_n}) \in \mathbb{R}^n$ for each $i = 1, 2, ..., n.$, Then the pair $(\mathbb{R}^n, \|.,...,.\|)$ is n-normed space. A complete n-normed space is said to be n-Banach space[3].

Let $(X, \|.,...,.\|)$ be an n-normed space of dimension $d \geq n \geq 2$ and $\{u_1, u_2, ..., u_n\}$ be a linearly independent set in X, A function $\|.,...,.\|_\infty$ on X^{n-1} defined by

$$\|x_1, ..., x_{n-1}\|_\infty = \max\{\|x_1, ..., x_n, u_i\|\}_{i=1}^n$$

is an $(n-1)$ norm on X with respect to $\{u_1, u_2, ..., u_n\}$.

A gauge on $[a, b]$ is a positive real-valued function $\delta : [a, b] \to \mathbb{R}^+$. This gauge is δ-fine if $[u_{i-1}, u_i] \subset [t_i - \delta(t_i), t_i + \delta(t_i)]$ while sequence of tagged partition P_n of [a,b] is a finite collection of ordered pairs $P_n = \{(x_{(i-1)_n}\, x_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ where $[x_{i-1}, x_i] \in [a, b]$, $x_{(i-1)_n} \leq t_{i_n} \leq x_{i_n}$ and $a = x_0 < x_{i_1} <, ..., < x_{m_n} = b$.

The following basic definitions of the Henstock Stieltjes integrable functions defined on a closed interval $[a, b] \subset \mathbb{R}$ were studied in [14], [16] and [20]. Also, interested readers can consult [19] for detailed theory on Sequential approach to Henstock integral.

Definition 2. *([15]) A function $f : [a, b] \to \mathbb{R}$ is Henstock integrable to A in $[a, b]$ if there exists a number $A \in \mathbb{R}$ such that for any $\varepsilon > 0$ there exists a function $\delta(x) > 0$ on $[a, b]$ such that for every $\delta(x) - fine$ partitions $P = \{(u_{i-1}, u_i), t_i\}_{i=1}^{n}$ of $[a, b]$, we have*

$$|S(f, P) - A| < \varepsilon.$$

where $S(f, P) = \sum_{i=1}^{n} f(t_i)(u_i - u_{(i-1)})$. We say that A is a Henstock integral of f on $[a, b]$ i.e $A = H \int_a^b f$. We use $H_f[a, b]$ to denote the set of all Henstock integrable functions defined on $[a, b]$.

We give the Sequential Henstock Stieltjes integral of real valued functions.

Definition 3. *([20]). Let $g : [a, b] \to \mathbb{R}$ be non decreasing function. A real valued function $f : [a, b] \to \mathbb{R}$ is Sequential Henstock Stieltjes integrable(SHS) with respect to g in $[a, b]$ if there exists a number $A \in \mathbb{R}$ and a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for every $\delta_n(x) - fine$ tagged partitions P_n, we have*

$$S(f, P_n) = \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] \to A, \qquad as \;\; n \to \infty.$$

We say that A is the Sequential Henstock Stieltjes integral of f on $[a, b]$ with $A = \int_{[a,b]} f dg$.

Definition 4. *([3]) Let X be a n-normed space and $\{x_n\}_{n=1}$ a sequence in X. We say that $\{x_n\}_{n=0}^{\infty}$ converge to same $x \in X$ if*

$$\lim_{n\to\infty} \|x_n - x, u_2, ..., u_n\| = 0$$

for all $u_2, ..., u_n \in X$.

We define newly the Sequential Henstock Stieltjes integrable function in an n-normed space X.

Definition 5. *Let X be an n-normed space and $g : [a, b] \subset X \to \mathbb{R}$ be non decreasing function. A real valued function f is Sequential Henstock Stieltjes integrable with respect to g if there exists a number $A \in X$ and a sequence of positive gauge functions $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for every $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}_{i=1}^{m_n}$ of $[a, b] \subset X$, we have*

$$\sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] \to A, \qquad as \;\; n \to \infty.$$

i.e.

$$\lim_{n\to\infty} \|S(f, P_n) - A, u_2, u_3, ..., u_n\| = 0,$$

for every $u_i \in X, (i = 2, ..., n)$, *where* $S(f, P_n) = \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})]$. *We say that* A *is the Sequential Henstock Stieltjes integral of* f *on* $[a, b]$ *with* $A = \int_{[a,b]\subset X} fdg$.

3. Main Results

We state and prove some of the simple properties of the Sequential Henstock Stieltjes integrable in n-normed space $(X, \|., ..., .\|)$.

Theorem 1. *(Uniqueness) Let* $g : [a, b] \subset X \to \mathbb{R}$ *be non decreasing function. If* f *is the Sequential Henstock Stieltjes integrable in* X*, then the integral is unique.*

Proof. Suppose that f is integrable with respect to g in X and let $A_1 = \int_{[a,b]\subset X} fdg$ and $A_2 = \int_{[a,b]\subset X} fdg$ with $A_1 \neq A_2$ where $A_1, A_2 \in \mathbb{R}$. Let $\varepsilon > 0$. Then there exists a sequence of positive gauge functions $\{\delta_n^1(x)\}$ and $\{\delta_n^2(x)\}$ in X such that for each $\delta_n^1(x)$-fine tagged partitions P_n^1 of $[a, b] \subset X$ and for each $\delta_n^2(x)$-fine tagged partitions P_n^1 and P_n^2 of $[a, b]$, we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - A_1, u_2, u_3, ..., u_n\| < \frac{\varepsilon}{2}$$

and

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - A_2, u_2, u_3, ..., u_n\| < \frac{\varepsilon}{2}$$

respectively, for every $u_2, u_3, ..., u_n \in X$.
Define a positive gauge function $\delta_n(x)$ on $[a, b] \subset X$ by $\delta_n(x) = \min\{\delta_n^1(x), \delta_n^2(x)\}$. Let P_n be any $\delta_n(x)$-fine tagged partition of $[a, b] \subset X$ and let $\varepsilon = \|A_1 - A_2\|$. Then

we have

$$\begin{aligned}\|A_1 - A_2\| &= (\|\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - A_1, u_2, u_3, ..., u_n \\ &\quad + \sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - A_2, u_2, u_3, ..., u_n\|) \\ &\leq \|\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - A_1, u_2, u_3, ..., u_n\| \\ &\quad +\|\sum_{i=1}^{m_n\in\mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - A_2, u_2, u_3, ..., u_n\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon = \|A_1 - A_2\|,\end{aligned}$$

which is a contradiction. And since for all $\varepsilon > 0$, there exists a $\delta_n(x) > 0$ on $[a, b]$, then $A_1 = A_2$.

We give the Cauchy criterion for the Sequential Henstock Stieltjes integrable function in an n-normed space.

Theorem 2. *(Cauchy Criterion). Let $g : [a, b] \subset X \to \mathbb{R}$ be non decreasing function. A function $f \in X$ is Sequential Henstock Stieltjes integrable with respect to g if and only if for every $\varepsilon > 0$, there exists a sequence of positive gauge functions $\{\delta_n(x)\}$ on $[a, b]$ such that for all $\delta_n(x) - fine$ tagged partitions P_n^1 and P_n^2, we have*

$$\|S(f, g, P_n^1) - S(f, g, P_n^2), u_2, u_3, ..., u_n\| < \varepsilon.$$

Proof. Suppose that $f \in X$ and $\int_{[a,b\subset X]} fdg = A$. Given $\varepsilon > 0$, there exists a $\{\delta_n(x)\}$ on $[a, b]$ such that

$$\|S(f, g, P_n^1) - A, u_2, u_3, ..., u_n\| < \frac{\varepsilon}{2}$$

and

$$\|S(f, g, P_n^2) - A, u_2, u_3, ..., u_n\| < \frac{\varepsilon}{2}$$

whenever P_n^1 and P_n^2 are $\delta_n(x) - fine$ tagged partitions of $[a, b]$ and for every $u_2, u_3, ..., u_n \in X$. Now,

$$\begin{aligned}\|S(f, g, P_n^1) - S(f, g, P_n^2), u_2, u_3, ..., u_n\| &\leq \|S(f, g, P_n^1) - A, u_2, u_3, ..., u_n\| \\ &\quad +\|S(f, g, P_n^2) - A, u_2, u_3, ..., u_n\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Thus, the Cauchy criterion is satisfied.
Suppose the Cauchy criterion holds. Then for each $m_n \in \mathbb{N}$, there exists a sequence of positive functions $\{\delta_n(x)\}_{n=1}^{\infty}$ on $[a,b]$ such that

$$\|S(f,g,Q_n^1) - S(f,g,Q_n^2), u_2, u_3, ..., u_n\| < \frac{1}{n}$$

whenever Q_n^1 and Q_n^2 are $\delta_n(x)$-fine tagged partitions of $[a,b]$. We now construct a Cauchy Sequence of Henstock sums which converges to real number denoted by A. Note without loss of generality, we may assume that $\{\delta_n(x)\}$ is a decreasing sequence for all $n \in \mathbb{N}$. Since P_n is $\delta_n(x)$-fine tagged partition of $[a,b]$, For $\varepsilon > 0$, by Archimedian property, there exists a natural number N such that $\frac{1}{\varepsilon} < N$. Let $N \leq n \leq m$, then P_n and P_m are $\delta_n(x)$-fine tagged partition of $[a,b]$, it follows that

$$\|S(f,g,P_n) - S(f,g,P_m), u_2, u_3, ..., u_n\| < \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Thus, $\{S(f,g,P_n)\}_{n=1}^{\infty}$ is a Cauchy Sequence. Hence the sequence $\{S(f,g,P_n)\}_{n=1}^{\infty}$ converges say to A as $n \to \infty$. Let $\varepsilon > 0$, there exists N_1 such that for all $n \geq N_1$, we have

$$\|S(f,g,P_n) - A, u_2, u_3, ..., u_n\| < \frac{\varepsilon}{2}.$$

By Archimedian principle, there exists N_2 such that $\dfrac{1}{\varepsilon} < N_2$. Now, let $N_0 = \max\{N_1, N_2\}$ and $n \geq N_0$. Let $\delta_n = \delta_{N_1}$, then for any $\varepsilon > 0$, there exists a $\delta_{N_1}(x) \in \{\delta_n(x)\}_{n=1}^{\infty}$ such that $n \geq N_1$ on $[a,b]$ and P_n, a sequence of $\delta_n(x)$-fine tagged partitions of $[a,b]$. Hence

$$\|S(f,g,P_n) - S(f,g,P_{N_1}), u_2, u_3, ..., u_n\| < \frac{1}{n} < \frac{1}{N_0} \leq \frac{1}{N_2} < \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned}
\|S(f,g,P_n) - A, u_2, u_3, ..., u_n\| &= \|S(f,g,P_n) - S(f,P_{N_1}), u_2, u_3, ..., u_n \\
&\quad + S(f,g,P_{N_1}) - A, u_2, u_3, ..., u_n\| \\
&\leq \|S(f,g,P_n) - S(f,g,P_{N_1}), u_2, u_3, ..., u_n\| \\
&\quad + \|S(f,g,P_{N_1}) - A, u_2, u_3, ..., u_n\| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon.
\end{aligned}$$

Theorem 3. *Let $g : [a, b] \subset X \to \mathbb{R}$ be non decreasing function. If $f \in X$, then f is Sequential Henstock Stieltjes integrable in X for every subinterval $[c, d]$ of $[a, b]$.*

Proof. Let $[c, d] \subseteq [a, b]$ and let $\varepsilon > 0$. Since f is Sequential Henstock Stieltjes integrable on $[a, b]$, By Cauchy Criterion, there exists a sequence of positive functions $\{\delta_n(x)\}_{n=1}^{\infty}$ on $[a, b]$ such that

$$\|S(f, g, P_n^1) - S(f, g, P_n^2), u_2, u_3, ..., u_n\| < \varepsilon.$$

whenever P_n^1 and P_n^2 are $\delta_n(x) - fine$ tagged partitions of $[a, b]$ and for every $u_2, u_3, ..., u_n \in X$. Consider the following cases.
Case 1. Suppose $a = c$. Let P_n^b be $\delta_n(x) - fine$ tagged partitions of $[d, b]$. Also $P_n^{1^*}$ and $P_n^{2^*}$ are $\delta_n(x) - fine$ tagged partitions of $[c, d]$. Define $P_n^1 = P_n^{1^*} \cup P_n^b$ and $P_n^2 = P_n^{2^*} \cup P_n^b$. Then P_n^1 and P_n^2 are $\delta_n(x) - fine$ tagged partitions of $[a, b]$. Hence,

$$\begin{aligned} \|S(f, g, P_n^{1^*}) - S(f, g, P_n^{2^*}), u_2, u_3, ..., u_n\| &= \|S(f, g, P_n^{1^*}) - S(f, g, P_n^b) + S(f, g, P_n^b) \\ &\quad -S(f, g, P_n^{2^*}), u_2, u_3, ..., u_n\| \\ &\leq \|S(f, g, P_n^1) - S(f, g, P_n^2), u_2, u_3, ..., u_n\| \\ &< \varepsilon. \end{aligned}$$

Case 2. Suppose $b = d$. The proof is similar to Case 1.

Case 3. Suppose $[c, d] \subset [a, b]$. The proof follows from Case 1 and Case 2.
Thus for any cases, f is Sequential Henstock Stieltjes integrable in n-normed space on every subinterval $[c, d]$ of $[a, b]$

Theorem 4. *Let $g : [a, b] \subset X \to \mathbb{R}$ be non decreasing function. If f is Sequential Henstock Stieltjes integrable in $[a, b]$, then f is Sequential Henstock Stieltjes integrable on $[a, c]$ and $[c, b]$ and*

$$\int_a^b f dg = \int_a^c f dg + \int_c^b f dg.$$

Proof. Suppose $\int_{[a,c]\subset X} f dg = A_1$ and $\int_{[c,b]\subset X} f dg = A_2$. Let $\varepsilon > 0$. Since f is Sequential Henstock Stieltjes integrable in $[a, c]$, there exists a sequence of positive functions$\{\delta_n^1(x)\}_{n=1}^{\infty}$ on $[a, c]$ such that $n_1 \geq n \in \mathbb{N}$ and $P_n^1 \ll \{\delta_n^1(x)\}_{n=1}^{\infty}$ implies

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - A_1, u_2, u_3, ..., u_n\| < \frac{\varepsilon}{2}$$

whenever P_n^1 is $\delta_n^1(x) - fine$ tagged partitions of $[a, c]$. Similarly, Since f is Sequential Henstock Stieltjes integrable in $[c, b]$, there exists a sequence of positive functions$\{\delta_n^2(x)\}_{n=1}^{\infty}$ on $[a, c]$ such that $n_2 \geq n$ and $P_n^2 \ll \{\delta_n^2(x)\}_{n=1}^{\infty}$ implies

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - A_2, u_2, u_3, ..., u_n\| < \frac{\varepsilon}{2}$$

whenever P_n^2 is $\delta_n^2(x) - fine$ tagged partitions of $[c, b]$ and for every $u_2, u_3, ..., u_n \in X$. Define a sequence of positive function $\{\delta_n(x)\}_{n=1}^{\infty}$ inorder to force the point c to be a tag of each $P_n^1 \ll \{\delta_n^1(x)\}$ for $n \geq N$. Using the right-left procedure in the gauge definition, we then split apart each partition P_n at the tag c so that it becomes a partition point of each P_n by

$$\delta_n(x) = \begin{cases} min\{\delta_n^1(x), (c - x)\}, & \text{if } a \leq x < c \\ min\{\delta_n^1(c), \delta_n^2(c)\}, & \text{if } x = c \\ min\{\delta_n^2(x), (x - c)\}, & \text{if } c < x \leq b. \end{cases}$$

Let $P_n \ll \delta_n(x) - fine$ for $n \geq N$. Let P_n^1 be $\delta_n^1(x) - fine$ tagged partitions of $[a, c]$ consisting $P_n \bigcap [a, c]$ and P_n^2 be $\delta_n^1(x) - fine$ tagged partitions of $[c, b]$ consisting $P_n \bigcap [c, b]$ for $n = 1, 2, 3, ...$ Then the right-left procedure provides that $S(f, g, P_n) = S(f, g, P_n^1) + S(f, P_n^2)$. Hence given $\varepsilon > 0$. there exists a $\{\delta_n(x)\}_{n=1}^{\infty}$ such that for $n \geq N$, we have

$$\| \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - (A_1 + A_2), u_2, u_3, ..., u_n\|$$

$$\begin{aligned} &= \| \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] + \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] \\ &\quad -(A_1 + A_2), u_2, u_3, ..., u_n\| \\ &\leq (\| \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - A_1, u_2, u_3, ..., u_n\| \\ &\quad +\| \sum_{i=1}^{m_n \in \mathbb{N}} f(t_{i_n})[g(u_{i_n}) - g(u_{(i-1)_n})] - A_2, u_2, u_3, ..., u_n\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

The following theorem shows the properties of linearity and integrability over interval $[a,b] \subset X$ of Sequential Henstock Stieltjes integrable functions in n-normed spaces. We omit the proofs which directly follow from the Definition.

Theorem 5. *Let $g : [a,b] \subset X \to \mathbb{R}$ be non decreasing function and $k \in \mathbb{R}$.*
i. If f is Sequential Henstock Stieltjes integrable in X then kf is Sequential Henstock Stieltjes integrable in X. Moreover,

$$\int_{[a,b]\subset X}(kf)dg = k\int_{[a,b]\subset X} fdg.$$

ii. If f is Sequential Henstock Stieltjes integrable in X and h is Sequential Henstock Stieltjes integrable in X, then $(f+h)$ is Sequential Henstock Stieltjes integrable in X . Moreover

$$\int_{[a,b]\subset X}(f+h)dg = \int_{[a,b]\subset X} fdg + \int_{[a,b]\subset X} hdg.$$

4. An Application

We discuss the following contraction condition mapping theorem.

Theorem 6. *Let $(X, \|.,...,.\|)$ be n-normed space, S be a self mapping of X with a fixed point ρ such that for all $x, y, u_1, ..., u_n \in X$ and for some $k \in [0,1)$, we have*

$$\|Sx - \rho, u_2, ..., u_n\| \leq k(\|x - \rho, u_2, ..., u_n\|). \tag{1}$$

Then S has a unique fixed point in X.

Proof. We shall establish that S has a unique fixed point by using condition (1): Suppose not. Then there exists $\rho^*, q^* \in F_S$ with $\rho^* \neq q^*$ and

$$\|\rho * -q^*, u_2, u_3, ..., u_n\| > 0.$$

Therefore, by (1), we have

$$\begin{aligned} 0 < \|\rho * -q^*, u_2, u_3, ..., u_n\| &= \|S\rho * -Sq^*, u_2, u_3, ..., u_n\| \\ &\leq k\|\rho * -q^*, u_2, u_3, ..., u_n\| \end{aligned}$$

(which is a contradiction), which leads to $1 - k > 0$ (since $k \in [0,1)$), but $\|\rho * -q^*, u_2, u_3, ..., u_n\| \leq 0$ (which is a contradiction).
Therefore, since n-normed is non negative, $\|\rho * -q^*, u_2, u_3, ..., u_n\| = 0$. i.e $\rho* = q^* = \rho$. Thus proving the uniqueness of the fixed points for S. Hence $F_s = \{\rho\}$

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