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UNIFIED COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS IN COMPLEX VALUED METRIC SPACES

W. M. Alfaqih

Abstract. The purpose of this manuscript is to prove some common fixed point theorems for two pairs of weakly compatible mappings in complex valued metric spaces satisfying an implicit relation. Some illustrative examples are also given which demonstrate the useful of our utilized implicit relation. Beside generalize and improve several well known conventional results of the existence literature, we obtain some new contractions which have not obtained before in complex valued metric spaces. As an application of our results, we prove the existence and uniqueness of common solution of Hammerstein integral equations and Urysohn integral equations.

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1. Introduction and preliminaries

In 1997, Popa [12] initiated the idea of an implicit relation which is designed to cover several well known contractions of the existing literature in one go besides admitting several new results. In fact, the strength of implicit relations lies in their unifying power besides being general enough to yield new contractions.

On the other hand, Azam et al. [1] introduced the concept of complex valued metric spaces which is more general than metric spaces and also proved common fixed point theorems for two mappings satisfying certain rational inequalities. Since then, several papers have dealt with fixed point theory in complex valued metric spaces (see [2], [4], [6], [14]-[19], [22] and references therein).

Let $\mathbb{C}$ be the set of all complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order $\precsim$ on $\mathbb{C}$ as follows:

$$z_1 \precsim z_2 \Longleftrightarrow Re(z_1) \leq Re(z_2) \wedge Im(z_1) \leq Im(z_2).$$

It follows that $z_1 \precsim z_2$, if on of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2),\ Im(z_1) = Im(z_2)$,

(ii) $Re(z_1) < Re(z_2),\ Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) = Re(z_2),\ Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) < Re(z_2),\ Im(z_1) < Im(z_2)$.

In particular, we write $z_1 = z_2$ if (i) holds and we write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (ii), (iii) and (iv) is satisfied while $z_1 \prec z_2$ if only (iv) is satisfied.

Throughout this work, $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}_+$ denote the set of all natural numbers, the set of all real numbers and the set of all $z \in \mathbb{C}$ such that $0 \precsim z$, respectively. Also, I denotes the identity mapping.

Remark 1. *Note that the following assertions hold for all $z_1, z_2, z_3 \in \mathbb{C}$:*

1. $\alpha, \beta \in \mathbb{R}$ *with* $\alpha \leq \beta$ *and* $0 \precsim z_1 \Longrightarrow \alpha z_1 \precsim \beta z_1$*;*
2. $0 \precsim z_1 \precnsim z_2 \Longrightarrow |z_1| < |z_2|$*;*
3. $z_1 \precsim z_2, z_2 \prec z_3 \Longrightarrow z_1 \prec z_3$*;*
4. *the dual relation of* $\precsim$ *is* $\succsim$.

The following basic definitions and results are required in the sequel.

Definition 1. *[1] Let X be a nonempty set. Suppose that the mapping $d : X \times X \longrightarrow \mathbb{C}_+$ satisfies the following conditions:*

(i) $d(x, y) = 0$ *if and only if* $x = y$ *for all* $x, y \in X$*;*

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$*;*

(iii) $d(x, y) \precsim d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Then the mapping d is called a complex valued metric and the pair (X, d) is called a complex valued metric space.

Remark 2. *In definition 1 we ignore stating the nonnegative property $0 \precsim d(x, y)$ for all $x, y \in X$ since it follows from $(i), (ii)$ and (iii).*

Definition 2. *[1] Let (X, d) be a complex valued metric space. Then*

1. *a point x in X is said to be an interior point of a set $M \subseteq X$, if there exists $0 \prec \varepsilon \in \mathbb{C}$ such that*

$$B(x, \varepsilon) = \{y \in X : d(x, y) \prec \varepsilon\} \subseteq M.$$

2. *a point x in X is called a limit point of a set $M \subseteq X$, if for every $0 \prec \varepsilon \in \mathbb{C}$,*
$$B(x,\varepsilon) \cap (M \setminus \{x\}) \neq \phi.$$
3. *a subset M of X is called an open set, if every element of M is an interior point of M.*
4. *a subset M of X is called a closed set, if every limit point of M belongs to M.*
5. *the family $F = \{B(x,\varepsilon) : x \in X, 0 \prec \varepsilon \in \mathbb{C}\}$ forms a subbasis of a Hausdorff topology τ on X.*

Example 1. *Let $X = C\big([a,b], \mathbb{R}^n\big)$ where $a, b \in \mathbb{R}, 0 < a \leq b$. Define a mapping $d : X \times X \longrightarrow \mathbb{C}$ as follows:*

$$d(x,y) = \max_{t \in [a,b]} \|x(t) - y(t)\|_\infty \sqrt{1+a^2} e^{i \arctan a}.$$

Then (X, d) is a complex valued metric space.

Definition 3. *[8] The* max *function for complex numbers with partial order relation $\precsim$ is defined as follows for all $z_1, z_2 \in \mathbb{C}$:*

$$\max\{z_1, z_2\} = z_2 \Longleftrightarrow |z_1| \leq |z_2|.$$

Definition 4. *[1] Let $\{x_n\}$ be a sequence in a complex valued metric space (X, d) and $x \in X$. Then*

(i) *$\{x_n\}$ is said to be a convergent and converges to x, If for every $0 \prec \varepsilon \in \mathbb{C}$ there exists an $n_0 \in \mathbb{N}$ such that*

$$d(x_n, x) \prec \varepsilon \quad \forall n > n_0$$

we denote this symbiotically by $\lim_{n\to\infty} x_n = x$ or $x_n \longrightarrow x$, as $n \longrightarrow \infty$.

(ii) *$\{x_n\}$ is said to be a Cauchy sequence if for every $0 \prec \varepsilon \in \mathbb{C}$ there exists an $n_0 \in \mathbb{N}$ such that*

$$d(x_n, x_{n+m}) \prec \varepsilon \quad \forall n > n_0$$

where $m \in \mathbb{N}$.

(iii) *(X, d) is called a complete complex valued metric space if every Cauchy sequence in X is convergent in X.*

Lemma 1. *[17] Let (X, d) be a complex valued metric space, $\{x_n\}$ a sequence in X and $\lambda \in [0, 1)$. If $\alpha_n = |d(x_n, x_{n+1})|$ satisfies $\alpha_n \leq \lambda\alpha_{n-1}$, for all $n \in \mathbb{N}$, then $\{x_n\}$ is Cauchy sequence.*

Definition 5. *Let S, T, f and g be four self-mappings of a nonempty set X, then*

(i) a point $u \in X$ is said to be a fixed point of S if $Su = u$,

(ii) a point $u \in X$ is said to be a common fixed point of S and T if $Su = Tu = u$,

(iii) a point $u \in X$ is said to be a coincidence point of S and f if $Su = fu$ and a point $t \in X$ such that $t = Su = fu$ is called a point of coincidence of S and f,

(iv) a pint $t \in X$ is said to be a common point of coincidence of the pairs (S, f) and (T, g) if there exist $u, v \in X$ such that $Su = fu = t$ and $Tv = gv = t$.

Definition 6. *Four families of self mappings $\{S_i\}_1^l, \{f_i\}_1^m, \{T_i\}_1^n$ and $\{g_i\}_1^s$, where $l, m, n, s \in \mathbb{N}$, are said to be pairwise commuting if:*

(i) $S_iS_j = S_jS_i$ for $i, j \in \{1, 2, ..., l\}$,

(ii) $S_if_j = f_jS_i$ for $i \in \{1, 2, ..., l\}, j \in \{1, 2, ..., m\}$,

(iii) $S_iT_j = T_jS_i$ for $i \in \{1, 2, ..., l\}, j \in \{1, 2, ..., n\}$,

(iv) $S_ig_j = g_jS_i$ for $i \in \{1, 2, ..., l\}, j \in \{1, 2, ..., s\}$,

(v) $f_if_j = f_jf_i$ for $i, j \in \{1, 2, ..., m\}$,

(vi) $f_iT_j = T_jf_i$ for $i \in \{1, 2, ..., m\}, j \in \{1, 2, ..., n\}$,

(vii) $f_ig_j = g_jf_i$ for $i \in \{1, 2, ..., lm\}, j \in \{1, 2, ..., s\}$,

(viii) $T_iT_j = T_jT_i$ for $i, j \in \{1, 2, ..., n\}$,

(ix) $T_ig_j = g_jT_i$ for $i \in \{1, 2, ..., n\}, j \in \{1, 2, ..., s\}$,

(x) $g_ig_j = g_jg_i$ for $i, j \in \{1, 2, ..., s\}$.

Remark 3. *On setting $f_i = g_j = I, \forall i \in \{1, 2, ..., m\}, j \in \{1, 2, ..., s\}$ in Definition 6 we get Definition (1.11) due to Imdad et al. [9].*

Definition 7. *[6] Let X be a complex valued metric space. A pair of self-mappings S, T on X is said to be weakly compatible if they commute at their coincidence points. i.e., $STx = TSx$ whenever $Sx = Tx, x \in X$.*

Definition 8. *[14] A function $f : \mathbb{C} \longrightarrow \mathbb{C}$ is said to be a lower semicontinuous at a point z_0 in $\mathbb{C}$ if for every $0 \prec \varepsilon \in \mathbb{C}$ there exists a neighborhood U of z_0 such that $f(z) \succsim f(z_0) - \varepsilon$ for all z in U. This can be expressed as $\liminf_{z\to z_0} f(z) \succsim f(z_0)$. Also, f is said to be an upper semicontinuous at a point z_0 in $\mathbb{C}$ if for every $0 \prec \varepsilon \in \mathbb{C}$ there exists a neighborhood U of z_0 such that $f(z) \precsim f(z_0) + \varepsilon$ for all z in U. This can be expressed as $\limsup_{z\to z_0} f(z) \precsim f(z_0)$.*

Definition 9. *[16] Let $T : \mathbb{C} \longrightarrow \mathbb{C}$ be a mapping. Then T is said to be a nondecreasing mapping with respect to $\precsim$ if for every $z_1, z_2 \in \mathbb{C}$, $z_1 \precsim z_2$ implies $Tz_1 \precsim Tz_2$.*

Definition 10. *[14] The control functions are defined as follows:*

(i) $\psi : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ is a continuous nondecreasing function with $\psi(z) = 0$ if and only if $z = 0$,

(i) $\phi : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ is a lower semicontinuous function with $\phi(z) = 0$ if and only if $z = 0$.

By Ψ and Φ we denote the set of all $\psi' s$ and the set of all $\phi' s$ respectively.

The aim of this manuscript is to utilize the idea of implicit relation in complex valued metric spaces to prove common fixed point results for two pairs of weakly compatible mappings satisfying an implicit relation such that these results unify, improve and generalize many existence results of the literature. We furnish with some examples to clarify that our implicit relation covers many of the exitance results in the context of complex valued metric spaces and also represent some new contraction conditions.

2. IMPLICIT RELATION

In this section, we extend the idea of implicit relation to complex valued metric spaces in order to prove unified complex metrical fixed point theorems.

Definition 11. *Let $\Im$ be the set of all complex valued lower semi-continuous functions $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ satisfying the following conditions:*

(F_1) *F is non-increasing in the fifth variable,*

(F_2) *there exists $h \in [0, 1)$ such that for $u, v \succsim 0$ with $F(u, v, v, u, u + v, 0) \precsim 0$ implies $u \precsim hv$,*

(F_3) *$F(u, 0, 0, u, u, 0) \succ 0$, $F(u, 0, u, 0, 0, u) \succ 0$ and $F(u, u, 0, 0, u, u) \succ 0$ for all $u \succ 0$.*

Example 2. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \lambda_1(z_2)z_2 - \lambda_2(z_2)\frac{z_4z_6}{1+z_2},$$

where $\lambda_1, \lambda_2 : \mathbb{C}_+ \longrightarrow [0, 1)$ *are given mappings.*
F_1 : *Obvious.*
F_2 : *Let* $u \succ 0$ *and* $F(u, v, v, u, u + v, 0) = u - \lambda_1(v)v \precsim 0$. *This implies that* $u \precsim \lambda_1(v)v \prec v$. *Hence there exists* $h \in [0, 1)$ *such that* $u \precsim hv$. *If* $u = 0$ *then it is clear.*
F_3 : *Let* $u \succ 0$, *then* $F(u, 0, 0, u, u, 0) = F(u, 0, u, 0, 0, u) = u \succ 0$ *and* $F(u, u, 0, 0, u, u) = u - \lambda_1(u)u \succ 0$. *Hence* $F \in \Im$.

Example 3. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = \psi(z_1) - \psi(\frac{z_4z_6}{1+z_2}) - \phi(\frac{z_4z_6}{1+z_2})$$

where $\psi \in \Psi$ *and* $\phi \in \Phi$.
F_1 : *Obvious.*
F_2 : $F(u, v, v, u, u + v, 0) = \psi(u) \precsim 0 \Rightarrow u = 0$. *Hence* (F_2) *holds trivially.*
F_3 : *Let* $u \succ 0$, *then*
$F(u, 0, 0, u, u, 0) = F(u, 0, u, 0, 0, u) = F(u, u, 0, 0, u, u) = \psi(u) \succ 0$. *Hence* $F \in \Im$.

Example 4. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = \psi(z_1) - \psi(\frac{z_4z_6}{1+z_2}) + \phi(\frac{z_4z_6}{1+z_2})$$

where $\psi \in \Psi$ *and* $\phi \in \Phi$.

Example 5. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = \psi(z_1) - \psi(\Delta) + \phi(\Delta)$$

where $\Delta \in \left\{t_2, t_3, \frac{t_6}{1+t_2}\right\}$, $\psi \in \Psi$ *with* $\psi(z_1) \precsim \psi(z_2) \Longleftrightarrow z_1 \precsim z_2$ *and* $\phi \in \Phi$.

If $\Delta = t_2$ *then* $F(t_1, t_2, t_3, t_4, t_5, t_6) = \psi(t_1) - \psi(t_2) + \phi(t_2)$,

- F_1 : *Obvious.*
- F_2 :*If* $u = 0$ *then it is clear. Assume that* $u \succ 0$ *and* $F(u, v, v, u, u + v, 0) = \psi(u) - \psi(v) + \phi(v) \precsim 0$. *Then we have*

$$\begin{aligned} \psi(u) &\precsim \psi(v) - \phi(v) \\ &\precsim \psi(v) \\ \Longrightarrow u &\precsim v \end{aligned}$$

If $u = v$*, then*

$$\begin{aligned} F(u,v,v,u,u+v,0) &= F(u,u,u,u,2u,0) \\ &= \phi(u) \precsim 0 \\ \Longrightarrow u &= 0 \end{aligned}$$

which is a contradiction. Thus $u \prec v$ *and hence there exists* $h \in [0,1)$ *such that* $u \precsim hv$*.*

- F_3 *: Let* $u \succ 0$*, then* $F(u,0,0,u,u,0) = F(u,0,u,0,0,u) = \psi(u) \succ 0$ *and* $F(u,u,0,0,u,u) = \phi(u) \succ 0$*. Hence* $F \in \Im$*.*

If $\Delta = t_3$ *then* $F(t_1,t_2,t_3,t_4,t_5,t_6) = \psi(t_1) - \psi(t_3) + \phi(t_3)$*,*

- F_1 *: Obvious.*
- F_2 *: Similar as in case* $\Delta = t_2$*.*
- F_3 *: Let* $u \succ 0$*, then* $F(u,0,0,u,u,0) = F(u,u,0,0,u,u) = \psi(u) \succ 0$ *and* $F(u,0,u,0,0,u) = \phi(u) \succ 0$*.*

Hence $F \in \Im$*.*
If $\Delta = \frac{t_6}{1+t_2}$ *then* $F(t_1,t_2,t_3,t_4,t_5,t_6) = \psi(t_1) - \psi(\frac{t_6}{1+t_2}) - \phi(\frac{t_6}{1+t_2})$*,*

- F_1 *: Obvious.*
- F_2 *:* $F(u,v,v,u,u+v,0) = \psi(u) \precsim 0 \Rightarrow u = 0$*. Thus* F_2 *holds trivially.*
- F_3 *: Let* $u \succ 0$*, then* $F(u,0,0,u,u,0) = \psi(u) \succ 0$*,* $F(u,0,u,0,0,u) = \phi(u) \succ 0$ *and* $F(u,0,0,u,u,0) = \psi(u) - \psi(\frac{u}{1+u}) + \phi(\frac{u}{1+u}) \succ 0$*. Since* $u \succ 0 \Rightarrow \frac{1}{1+u} \prec 1 \Rightarrow \psi\left(\frac{u}{1+u}\right) \precsim \psi(u) \Rightarrow \psi(u) - \psi\left(\frac{u}{1+u}\right) \succsim 0$ *and* $\phi\left(\frac{u}{1+u}\right) \succ 0$*. Hence* $F \in \Im$*.*

Example 6. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1,z_2,z_3,z_4,z_5,z_6) = \begin{cases} z_1 - \lambda z_2 - \mu\frac{z_3 z_4}{z_4+z_6} - \gamma\frac{z_5 z_6}{z_4+z_6}, & if\ z_4 + z_6 \neq 0; \\ z_1, & if\ z_4 + z_6 = 0. \end{cases}$$

where $\lambda, \mu, \gamma \in \mathbb{R}_+$ *such that* $\lambda + \mu < 1$ *and* $\lambda + \gamma < 1$*.*
If $F(z_1,z_2,z_3,z_4,z_5,z_6) = z_1 - \lambda z_2 - \mu\frac{z_3 z_4}{z_4+z_6} - \gamma\frac{z_5 z_6}{z_4+z_6}$*, then*
F_1 *: Obvious.*
F_2 *: Let* $u \succ 0$*, then*

$$\begin{aligned} F(u,v,v,u,u+v,0) &= u - \lambda v - \mu\frac{vu}{u} \precsim 0 \\ \Longrightarrow u &\precsim (\lambda+\mu)v \\ &\prec v. \end{aligned}$$

Hence there is $h \in [0,1)$ *such that* $u \precsim hv$. *If* $u = 0$, *then it is clear.*
F_3 : *Let* $u \succ 0$, *then* $F(u,0,0,u,u,0) = F(u,0,u,0,0,u) = u \succ 0$. *Also* $F(u,u,0,0,u,u) = u - \lambda u - \gamma \frac{u^2}{u} = u - \lambda u - \gamma u \succ 0$. *Hence* $F \in \Im$.

Example 7. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1,z_2,z_3,z_4,z_5,z_6) = \begin{cases} z_1 - \lambda z_2 \frac{z_3+z_4}{z_2+z_4} - \mu z_2 \frac{z_5+z_6}{z_2+z_4}, & if z_2 + z_4 \neq 0; \\ z_1, & if z_2 + z_4 = 0. \end{cases}$$

where $\lambda, \mu \in \mathbb{R}_+$ *such that* $\lambda + 2\mu < 1$.

Example 8. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1,z_2,z_3,z_4,z_5,z_6) = \begin{cases} z_1 - \alpha z_3 - \beta \frac{z_2 z_5}{z_2+z_4} - \gamma \frac{z_4 z_6}{z_2+z_4} - \delta \frac{z_2 z_6}{z_2+z_4} & \\ \qquad -\eta \frac{z_4 z_5}{z_2+z_4}, & if z_2 + z_4 \neq 0; \\ z_1, & if z_2 + z_4 = 0. \end{cases}$$

where $\alpha, \beta, \gamma, \delta, \eta \in \mathbb{R}_+$ *such that* $\alpha + \beta + \eta < 1$ *and* $\beta + \delta < 1$.

Example 9. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1,z_2,z_3,z_4,z_5,z_6) = \begin{cases} z_1 - \alpha z_3 - \beta \frac{z_2 z_5}{z_3+z_4+z_6}, & if \ z_3 + z_4 + z_6 \neq 0; \\ z_1, & if \ z_3 + z_4 + z_6 = 0. \end{cases}$$

where $\alpha, \beta \in \mathbb{R}_+$ *such that* $\alpha + \beta < 1$.

Example 10. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$\begin{aligned} F(z_1,z_2,z_3,z_4,z_5,z_6) = z_1 - \alpha_1 z_2 - \alpha_2(z_3+z_4) - \alpha_3(z_5+z_6) - \alpha_4 \frac{z_4(1+z_3)}{1+z_2} \\ - \alpha_5 \frac{z_2(1+z_3+z_6)}{1+z_2} - \alpha_6 z_5, \end{aligned}$$

where $\alpha_i \in \mathbb{R}_+, i = 1,2,...,6,$ *such that* $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + 2\alpha_6 < 1$.

Example 11. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1,z_2,z_3,z_4,z_5,z_6) = \begin{cases} z_1 - \lambda \frac{z_2 z_6 + z_4 z_5}{z_2+z_4+z_6}, & if z_2 + z_4 + z_6 \neq 0; \\ z_1, & if z_2 + z_4 + z_6 = 0. \end{cases}$$

where $\lambda \in \mathbb{R}_+$ *such that* $\lambda < 1$.

Example 12. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = \begin{cases} z_1 - \alpha_1 z_2 - \alpha_2 \frac{z_3 z_4}{z_2+z_6} - \alpha_3 \frac{z_5 z_2}{z_2+z_6} - \alpha_4 \frac{z_3 z_2}{z_2+z_6}, & if z_2 + z_6 \neq 0; \\ z_1, & if z_2 + z_6 = 0. \end{cases}$$

,

where $\alpha_i \in \mathbb{R}_+, i = 1, 2, 3, 4$ *such that* $\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 < 1$.

Example 13. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = \begin{cases} z_1 - \alpha_1 z_3 - \alpha_2 (z_2 + z_5) - \alpha_3 (z_4 + z_6) & \\ \qquad - \alpha_4 \frac{z_3 z_5 + z_4 z_6}{z_2 + z_4}, & if z_2 + z_4 \neq 0; \\ z_1, & if z_2 + z_4 = 0. \end{cases}$$

where $\alpha_i \in \mathbb{R}_+, i = 1, 2, 3, 4$ *such that* $\alpha_1 + 3\alpha_2 + \alpha_3 + \alpha_4 < 1$.

Example 14. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha z_3 - \beta \max\{z_2, z_3, z_6\} - \gamma \max\{z_3, z_5\},$$

where $\alpha, \beta, \gamma \in \mathbb{R}_+$ *such that* $\alpha + \beta + 2\gamma < 1$.

Example 15. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max\{z_2, z_3, z_4, z_5, z_6\}, \alpha \in [0, \frac{1}{2}).$$

Example 16. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = \alpha_1 z_1 - \alpha_2 z_2 - \alpha_3 z_3 - \alpha_4 z_4 - \alpha_5 z_5 - \alpha_6 z_6,$$

where $\alpha_i \in \mathbb{R}_+, i = 1, 2, ..., 6$ *such that* $\alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6 < \alpha_1$ *and* $\alpha_1 > 0$.

Example 17. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max\{z_3 + z_4, z_5 + z_6\}, \alpha \in [0, \frac{1}{2}).$$

Example 18. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max\{z_2, z_3, z_4, z_5 + z_6\}, \alpha \in [0, \frac{1}{2}).$$

Example 19. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max\{z_2, z_3, \ z_5, z_4 + z_6\}, \alpha \in [0, \frac{1}{2}).$$

Example 20. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha(z_4 + z_6) - \alpha \max\{z_2, z_3, z_5\}, \alpha \in [0, \frac{1}{3}).$$

Example 21. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max \left\{\frac{2z_2 + z_5}{2}, \frac{2z_2 + z_4}{2}, \frac{2z_2 + z_6}{2}\right\}, \alpha \in [0, \frac{1}{2}).$$

Example 22. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max \left\{z_3, \frac{z_2 + z_5}{2}, \frac{z_4 + z_6}{2}\right\}, \alpha \in [0, \frac{2}{3}).$$

Example 23. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1^2 - \alpha z_1(z_2 + z_3 + z_5) - \beta z_4 z_6.$$

where $\alpha, \beta \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{4}$.

Example 24. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1^2 - z_1(\alpha_1 z_2 + \alpha_2 z_3 + \alpha_3 z_4) - \alpha_4 z_5 z_6.$$

where $\alpha_i \in \mathbb{R}_+, i = 1, 2, 3, 4$ *such that* $\alpha_1 + \alpha_2 + \alpha_3 < 1$ *and* $\alpha_1 + \alpha_4 < 1$.

Example 25. *Define a function* $F : \mathbb{C}_+^6 \longrightarrow \mathbb{C}$ *as follows:*

$$F(z_1, z_2, z_3, z_4, z_5, z_6) = z_1 - \alpha \max\{z_2, z_3, z_4, z_5\} - \beta[z_5 + z_6]$$

where $\alpha, \beta \in \mathbb{R}_+$ *such that* $\alpha + \beta < \frac{1}{2}$.

3. MAIN RESULTS

Now, we present the main results of this paper.

Theorem 2. *Let* S, T, f *and* g *be four self-mappings on a complex valued metric space* (X, d). *Assume that there exists* $F \in \Im$ *such that for all* $x, y \in X$,

$$F(d(Sx, Ty), d(fx, gy), d(Sx, fx), d(Ty, gy), d(Ty, fx), d(Sx, gy)) \precsim 0 \qquad (1)$$

If $SX \subseteq gX$, $TX \subseteq fX$ *and* $gX \cup fX$ *is complete, then the pairs* (S, f) *and* (T, g) *have a unique common point of coincidence.*

Moreover, if the pairs (S, f) *and* (T, g) *are weakly compatible, then* S, T, f *and* g *have a unique common fixed point in* X.

Proof. Let x_0 be an arbitrary point in X . Since $SX \subseteq gX$, we can find a point x_1 in X such that $Sx_0 = gx_1$. Also, since $TX \subseteq fX$, we can choose a point x_2 in X with $Tx_1 = fx_2$. Thus, in general for the point x_{2n} one can find a point x_{2n+1} such that $Sx_{2n} = gx_{2n+1}$ and also a point x_{2n+2} with $Tx_{2n+1} = fx_{2n+2}$ for $n = 0, 1, 2, ...$. Repeating such arguments we can construct two sequences $\{x_n\}$ and $\{y_n\}$ by the rule

$$Sx_{2n-2} = gx_{2n-1} = y_{2n-1} \quad and \quad Tx_{2n-1} = fx_{2n} = y_{2n} \quad , n = 1, 2, 3, ... \tag{2}$$

Clearly $\{y_n\} \subseteq gX \cup fX$. We will prove that $\{y_n\}$ is a Cauchy sequence in $gX \cup fX$.

Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (1), we have

$$F(d(Sx_{2n}, Tx_{2n+1}), d(fx_{2n}, gx_{2n+1}), d(Sx_{2n}, fx_{2n}), d(Tx_{2n+1}, gx_{2n+1}) \\ , d(Tx_{2n+1}, fx_{2n}), d(Sx_{2n}, gx_{2n+1})) \precsim 0 \tag{3}$$

On using (2) and (3), we have

$$F(d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \\ d(y_{2n+2}, y_{2n}), 0) \precsim 0$$

Now, due to (F_1) and triangle inequality, we get that

$$F(d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n}), d(y_{2n+2}, y_{2n+1}), \\ d(y_{2n+2}, y_{2n+1}) + d(y_{2n+1}, y_{2n}), 0) \precsim 0$$

implying thereby $d(y_{2n+1}, y_{2n+2}) \precsim hd(y_{2n}, y_{2n+1})$ (due to F_2). Similarly, by taking $x = x_{2n-1}$ and $y = x_{2n}$ in (1),one can prove that $d(y_{2n}, y_{2n+1}) \precsim hd(y_{2n-1}, y_{2n})$. Thus $d(y_n, y_{n+1}) \precsim hd(y_{n-1}, y_n) \ \ \forall n \in \mathbb{N} - \{1\}$ which implies that $|d(y_n, y_{n+1})| \leq h|d(y_{n-1}, y_n)| \ \ \forall n \in \mathbb{N} - \{1\}$.
Hence by Lema (1), $\{y_n\}$ is a Cauchy sequence in $gX \cup fX$. Since $gX \cup fX$ is complete it follows that $\{y_n\}$ converges to some $t \in gX \cup fX$. Therefor in the light of (2), one can get

$$\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n} = \lim_{n\to\infty} gx_{2n+1} = t \tag{4}$$

Now, if $t \in gX$, then there exists $u \in X$ such that $gu = t$. We will prove that $Tu = t$. On contrary, assume that $d(Tu, t) \succ 0$. Putting $x = x_{2n}$ and $y = u$ in (1), we have

$$F(d(Sx_{2n}, Tu), d(fx_{2n}, gu), d(Sx_{2n}, fx_{2n}), d(Tu, gu), \\ d(Tu, fx_{2n}), d(Sx_{2n}, gu)) \precsim 0$$

Taking $n \longrightarrow \infty$ and using (4), we obtain

$$F(d(t,Tu), d(t,gu), 0, d(Tu,gu), d(Tu,t), d(t,gu)) \precsim 0$$

Since $gu = t$, we get that

$$F(d(t,Tu), 0, 0, d(Tu,t), d(Tu,t), 0) \precsim 0,$$

which is a contradiction to F_3. Hence $Tu = t$. Therefor, we have

$$Tu = gu = t, \tag{5}$$

proving that t is a point of coincidence of the pair (S, f).

Since $TX \subseteq fX$, there exists $v \in X$ such that $fv = t$. Setting $x = v$ and $y = x_{2n+1}$ in (1), in similar argument one can prove that

$$Sv = fv = t \tag{6}$$

That is, t is also a point of coincidence of the pair (T, g). Hence t is a common point of coincidence of (S, f) and (T, g).

Now, we prove that t is unique. Let t' be a point of coincidence of both (S, f) and (T, g) such that $d(t,t') \succ 0$. Then there exist $u', v' \in X$ such that $Su' = fu' = t'$ and $Tv' = gv' = t'$. Setting $x = u'$ and $y = v$ in (1), we have

$$\begin{aligned} F(d(Su',Tv), d(fu',gv), d(Su',fu'), d(Tv,gv), d(Tv,fu'),& \\ d(Su',gv) \quad &\precsim \quad 0, \end{aligned}$$

this implies that $F(d(t',t), d(t',t), 0, 0, d(t,t'), d(t',t) \precsim 0$, which is a contradiction to (F_3). Therefor, (S, f) and (T, g) have unique point of coincidence.

Now, on using (5), (6) and the weak compatibility of the pairs (S, f) and (T, g), we have

$$St = Sfv = fSv = ft, \tag{7}$$

$$Tt = Tgu = gTu = gt, \tag{8}$$

i.e, t is a coincidence point of each one of the pairs (S, f) and (T, g).

Next, we show that t is a common fixed point of S, T, f and g. First we show that $St = t$. If not, then $d(St,t) \succ 0$. Setting $x = t$ and $y = u$ in (1), we have

$$F(d(St,Tu), d(ft,gu), d(St,ft), d(Tu,gu), d(Tu,ft), d(St,gu) \precsim 0,$$

Using (5) and (7),we obtain

$$F(d(St,t),d(St,t),0,0,d(t,St),d(St,t)\precsim 0,$$

which is a contradiction to (F_3). Thus

$$St=ft=t \tag{9}$$

Similarly, one can prove that

$$Tt=gt=t \tag{10}$$

On combining (9) and (10), we have

$$St=Tt=ft=gt=t$$

i.e, t is a common fixed point of S,T,f and g.

Finally, we show that t is unique. Assume that t^* is another common fixed point of S,T,f and g. This implies that t^* is a common point of coincidence of the pairs (S,f) and (T,g) which is a contradiction to the fact that t is unique. Hence t is unique common fixed point of S,T,f and g. The proof is similar in case $t\in fX$. This completes the proof.

Remark 4. *Theorem 2 generalizes and improves Theorem (2) of Popa [13].*

As a consequence of Theorem 2, we have the following theorem for four finite families of self mappings defined on a complex valued metric space which can be viewed as a generalization to Theorem (2.2) of M.Imdad et al.[7].

Theorem 3. *Let $\{S_i\}_1^l,\{T_j\}_1^n,\{f_k\}_1^m$ and $\{g_r\}_1^s$ be four finite pairwise commuting families of self-mappings defined on a complex valued metric space (X, d). Let $S=S_1S_2...S_l, T=T_1T_2...T_n, f=f_1f_2...f_m$ and $g=g_1g_2...g_s$ satisfy inequality (1), $SX\subseteq gX$, $TX\subseteq fX$ and $gX\cup fX$ is complete subspace of X. Then*

- *(a) the pairs (S,f) and (T,g) have a unique common point of coincidence,*
- *(b) S,T,f and g have a unique common fixed point,*
- *(c) the component maps of the families $\{S_i\}_1^l,\{T_j\}_1^m,\{f_k\}_1^n$ and $\{g_r\}_1^s$ have a unique common fixed point.*

Proof. By the componentwise commutativity of the pairs $\left(\{S_i\}_1^l, \{f_k\}_1^m\right)$ and $\left(\{T_j\}_1^n, \{g_r\}_1^s\right)$, one can prove that $Sf = fS$ and $Tg = gT$ and hence, the pairs (S, f) and (T, g) are weak compatible.

Clearly conditions of Theorem (1) are satisfied for S, T, f and g. Thus (S, f) and (T, g) have a unique common point of coincidence which established (a) and also S, T, f and g have a unique common fixed point $t \in X$ which established (b). Now, we show that t is also a common fixed point of the component maps of the families $\{S_i\}_1^l, \{T_j\}_1^m, \{f_k\}_1^n$ and $\{g_r\}_1^s$. For this consider

$$\begin{aligned} S(S_i t) &= (SS_i)t \\ &= (S_1 S_2 ... S_l S_i)t \\ &= (S_1 S_2 ... S_i S_l)t \\ &= (S_1 S_i ... S_l)t \\ &= (S_i S_1 ... S_l)t \\ &= (S_i S)t \\ &= S_i(St) \\ &= S_i t \ , i \in \{1, 2, ..., l\} \end{aligned}$$

Similarly, one can prove that $S(T_j t) = T_j t, S(f_k t) = f_k t, S(g_r t) = g_r t, T(T_j t) = T_j t, T(S_i t) = S_i t, T(f_k t) = f_k t, T(g_r t) = g_r t, f(f_k t) = f_k t, f(S_i t) = S_i t, f(T_j t) = T_j t, f(g_r t) = g_r t, g(g_r t) = g_r t, g(S_i t) = S_i t, g(T_j t) = T_j t, g(f_k t) = f_k t$, for all $i \in \{1, 2, ..., l\}, j \in \{1, 2, ..., m\}, k \in \{1, 2, ..., n\}$ and $r \in \{1, 2, ..., s\}$. Therefor $S_i t, T_j t, f_k t$ and $g_r t$ are also fixed points of S, T, f and g. But according to Theorem 2 we have that the common fixed point of S, T, f and g is unique and hence (for all i, j, k and r) one gets

$$S_i t = T_j t = f_k t = g_r t = t$$

proving that t is a common fixed point of $S_i t, T_j t, f_k t$ and $g_r t$ for all i, j, k and r. Finally, we observe that t is unique common fixed point of $S_i t, T_j t, f_k t$ and $g_r t$ for all i, j, k and r. Otherwise, let t^* another common fixed point of $S_i t, T_j t, f_k t$ and $g_r t$ for all i, j, k and r. Then one can prove that t^* is also a common fixed point of S, T, f and g which is a contradiction. This completes the proof.

By setting $S_i = S, T_j = T, f_k = f, g_r = g$, for all i, j, k and r, in Theorem 3 one can deduce the following corollary which can be viewed as a generalization of Theorem 2.

Corollary 4. *Let* (X,d) *be a complex valued metric space and* S,T,f *and* g *be four self-mappings on* X. *Assume that there exists* $F \in \Im$ *such that for all* $x,y \in X$,

$$F(d(S^l x, T^m y), d(f^n x, g^s y), d(S^l x, f^n x), d(T^m y, g^s y), d(T^m y, f^n x), d(S^l x, g^s y)) \precsim 0, \quad (11)$$

where $l,m,n,s \in \mathbb{N}$.

If $S^l X \subseteq g^s X$, $T^m X \subseteq f^n X$ *and* $g^s X \cup f^n X$ *is complete, then*

(a) the pairs (S,f) *and* (T,g) *have a unique common point of coincidence,*

(b) S,T,f *and* g *have a unique common fixed point.*

In view of Examples 2-25 we have the following corollaries which cover, generalize and improve several known results as well as give new contraction conditions in complex valued metric spaces.

Corollary 5. *The conclusions of Theorem (2) remain true if for all* $x,y \in X$ *the implicit relation (1) is replaced by one of the following:*

(a_1*)*

$$\begin{aligned} d(Sx,Ty) &\precsim \lambda_1(d(fx,gy))d(fx,gy) \\ &+ \lambda_2(d(fx,gy))\frac{d(Ty,gy)d(Sx,gy)}{1+d(fx,gy)}, \end{aligned}$$

where $\lambda_1, \lambda_2 : \mathbb{C}_+ \longrightarrow [0,1)$ *are given mappings.*

(a_2*)*

$$\psi(d(Sx,Ty)) \precsim \psi\left(\frac{d(Ty,gy)d(Sx,gy)}{1+d(fx,gy)}\right) + \phi\left(\frac{d(Ty,gy)d(Sx,gy)}{1+d(fx,gy)}\right)$$

where $\psi \in \Psi$ *and* $\phi \in \Phi$.

(a_3*)*

$$\psi(d(Sx,Ty)) \precsim \psi\left(\frac{d(Ty,gy)d(Sx,gy)}{1+d(fx,gy)}\right) - \phi\left(\frac{d(Ty,gy)d(Sx,gy)}{1+d(fx,gy)}\right)$$

where $\psi \in \Psi$ *and* $\phi \in \Phi$.

(a_4*)*

$$\psi\big(d(Sx,Ty)\big) \precsim \psi\big(\Delta\big) - \phi\big(\Delta\big)$$

where $\Delta \in \left\{ d(fx,gy), d(Sx,fx), \frac{d(Sx,gy)}{1+d(fx,gy)} \right\}$, $\psi \in \Psi$ *with* $\psi(z_1) \precsim \psi(z_2) \Longleftrightarrow z_1 \precsim z_2$ *and* $\phi \in \Phi$.

(a_5)

$$d(Sx,Ty) \precsim \begin{cases} \lambda d(fx,gy) + \mu \frac{d(Sx,fx)d(Ty,gy)}{d(Ty,gy)+d(Sx,gy)} \\ \qquad +\gamma \frac{d(Ty,fx)d(Sx,gy)}{d(Ty,gy)+d(Sx,gy)}, & if d(Ty,gy)+d(Sx,gy) \neq 0; \\ 0, & if d(Ty,gy)+d(Sx,gy) = 0. \end{cases}$$

where $\lambda, \mu, \gamma \in \mathbb{R}_+$ such that $\lambda + \mu < 1$ and $\lambda + \gamma < 1$.

(a_6)

$$d(Sx,Ty) \precsim \begin{cases} \lambda d(fx,gy) \frac{d(Sx,fx)+d(Ty,gy)}{d(fx,gy)+d(Ty,gy)} \\ \qquad +\mu d(fx,gy) \frac{d(Ty,fx)+d(Sx,gy)}{d(fx,gy)+d(Ty,gy)}, & if d(fx,gy)+d(Ty,gy) \neq 0; \\ 0, & if d(fx,gy)+d(Ty,gy) = 0. \end{cases}$$

where $\lambda, \mu \in \mathbb{R}_+$ such that $\lambda + 2\mu < 1$.

(a_7)

$$d(Sx,Ty) \precsim \begin{cases} \alpha d(Sx,fx) + \beta \frac{d(fx,gy)d(Ty,fx)}{d(fx,gy)+d(Ty,gy)} \\ +\gamma \frac{d(Ty,gy)d(Sx,gy)}{d(fx,gy)+d(Ty,gy)} + \delta \frac{d(fx,gy)d(Sx,gy)}{d(fx,gy)+d(Ty,gy)} \\ +\eta \frac{d(Ty,gy)d(Ty,fx)}{d(fx,gy)+d(Ty,gy)}, & if d(fx,gy)+d(Ty,gy) \neq 0; \\ 0, & if d(fx,gy)+d(Ty,gy) = 0. \end{cases}$$

where $\alpha, \beta, \gamma, \delta, \eta \in \mathbb{R}_+$ such that $\alpha + \beta + \eta < 1$ and $\beta + \delta < 1$.

(a_8)

$$d(Sx,Ty) \precsim \begin{cases} \alpha d(Sx,fx) \\ +\beta \frac{d(fx,gy)d(Ty,fx)}{d(Sx,fx)+d(Ty,gy)+d(Sx,gy)}, & if d(Sx,fx)+d(Ty,gy)+d(Sx,gy) \neq 0; \\ 0, & if d(Sx,fx)+d(Ty,gy)+d(Sx,gy) = 0. \end{cases}$$

where $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha + \beta < 1$.

(a_9)

$$\begin{aligned} d(Sx,Ty) \precsim{}& \alpha_1 d(fx,gy) + \alpha_2[d(Sx,fx)+d(Ty,gy)] \\ &+ \alpha_3[d(Ty,fx)+d(Sx,gy)] + \alpha_4 \frac{d(Ty,gy)[1+d(Sx,fx)]}{1+d(fx,gy)} \\ &+ \alpha_5 \frac{d(fx,gy)[1+d(Sx,fx)+d(Sx,gy)]}{1+d(fx,gy)} + \alpha_6 d(Ty,fx), \end{aligned}$$

where $\alpha_i \in \mathbb{R}_+, i = 1,2,...,7,$ such that $\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + 2\alpha_6 < 1$.

(a_{10})

$$d(Sx,Ty) \precsim \begin{cases} \lambda \frac{d(fx,gy)d(Sx,gy)+d(Ty,gy)d(Ty,fx)}{d(fx,gy)+d(Ty,gy)+d(Sx,gy)}, & ifd(fx,gy)+d(Ty,gy)+d(Sx,gy) \neq 0; \\ 0, & ifd(fx,gy)+d(Ty,gy)+d(Sx,gy) = 0. \end{cases}$$

where $\lambda \in \mathbb{R}_+$ such that $\lambda < 1$.

(a_{11})

$$d(Sx,Ty) \precsim \begin{cases} \alpha_1 d(fx,gy) + \alpha_2 \frac{d(Sx,fx)d(Ty,gy)}{d(fx,gy)+d(Sx,gy)} & \\ +\alpha_3 \frac{d(Ty,fx)d(fx,gy)}{d(fx,gy)+d(Sx,gy)} + \alpha_4 \frac{d(Sx,fx)d(fx,gy)}{d(fx,gy)+d(Sx,gy)}, & ifd(fx,gy)+d(Sx,gy) \neq 0; \\ 0, & ifd(fx,gy)+d(Sx,gy) = 0. \end{cases}$$

where $\alpha_i \in \mathbb{R}_+, i = 1,2,3,4$ such that $\alpha_1+\alpha_2+2\alpha_3+\alpha_4 < 1$.

(a_{12})

$$d(Sx,Ty) \precsim \begin{cases} \alpha_1 d(Sx,fx) + \alpha_2[d(fx,gy)+d(Ty,fx)] & \\ +\alpha_3[d(Ty,gy)+d(Sx,gy)] & \\ +\alpha_4 \frac{d(Sx,fx)d(Ty,fx)+d(Ty,gy)d(Sx,gy)}{d(fx,gy)+d(Ty,gy)}, & ifd(fx,gy)+d(Ty,gy) \neq 0; \\ 0, & d(fx,gy)+d(Ty,gy) = 0. \end{cases}$$

where $\alpha_i \in \mathbb{R}_+, i = 1,2,3,4$ such that $\alpha_1+3\alpha_2+\alpha_3+\alpha_4 < 1$.

(a_{13})

$$d(Sx,Ty) \precsim \alpha d(Sx,fx) + \beta \max\{d(fx,gy), d(Sx,fx), d(Sx,gy)\} \\ + \gamma \max\{d(Sx,fx), d(Ty,fx)\},$$

where $\alpha, \beta, \gamma \in \mathbb{R}_+$ such that $\alpha+\beta+2\gamma < 1$.

(a_{14})

$$d(Sx,Ty) \precsim \alpha \max\{d(fx,gy), d(Sx,fx), d(Ty,gy), d(Ty,fx), d(Sx,gy)\},$$

where $\alpha \in \mathbb{R}_+$ such that $\alpha < \frac{1}{2}$.

(a_{15})

$$\alpha_1 d(Sx,Ty) \precsim \alpha_2 d(fx,gy)+\alpha_3 d(Sx,fx)+\alpha_4 d(Ty,gy)+\alpha_5 d(Ty,fx)+\alpha_6 d(Sx,gy),$$

where $\alpha_i \in \mathbb{R}_+, i = 1,2,...,6$ such that $\alpha_2+\alpha_3+\alpha_4+2\alpha_5+\alpha_6 < \alpha_1$ and $\alpha_1 > 0$.

(a_{16})

$$d(Sx,Ty) \precsim \alpha \max\{d(Sx,fx)+d(Ty,gy), d(Ty,fx)+d(Sx,gy)\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{2}$.

(a_{17})

$$d(Sx,Ty) \precsim \alpha \max\ \{d(fx,gy), d(Sx,fx), d(Ty,gy), d(Ty,fx)+d(Sx,gy)\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{2}$.

(a_{18})

$$d(Sx,Ty) \precsim \alpha \max\{d(fx,gy), d(Sx,fx), d(Ty,fx), d(Ty,gy)+d(Sx,gy)\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{2}$.

(a_{19})

$$d(Sx,Ty) \precsim \alpha[d(Ty,gy)+d(Sx,gy)]+\alpha \max\{d(fx,gy), d(Sx,fx), d(Ty,fx)\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{3}$.

(a_{20})

$$d(Sx,Ty) \precsim \alpha \max\left\{\frac{2d(fx,gy)+d(Ty,fx)}{2}, \frac{2d(fx,gy)+d(Ty,gy)}{2}, \frac{2d(fx,gy)+d(Sx,gy)}{2}\right\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{2}$.

(a_{21})

$$d(Sx,Ty) \precsim \alpha \max\ \left\{d(Sx,fx), \frac{d(fx,gy)+d(Ty,fx)}{2}, \frac{d(Ty,gy)+d(Sx,gy)}{2}\right\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{2}{3}$.

(a_{22})

$$d^2(Sx,Ty) \precsim \alpha d(Sx,Ty)\Big[d(fx,gy)+d(Sx,fx)+d(Ty,fx)\Big]+\beta d(Ty,gy)d(Sx,gy),$$

where $\alpha, \beta \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{4}$.

(a_{23})

$$d^2(Sx,Ty) \precsim d(Sx,Ty)\Big[\alpha_1 d(fx,gy)+\alpha_2 d(Sx,fx)+\alpha_3 d(Ty,gy)\Big]+\alpha_4 d(Ty,fx)d(Sx,gy),$$

where $\alpha_i \in \mathbb{R}_+, i=1,2,3,4$ *such that* $\alpha_1+\alpha_2+\alpha_3<1$ *and* $\alpha_1+\alpha_4<1$.

(a_{24})

$$\begin{aligned} d(Sx,Ty) \precsim \alpha \max\{d(fx,gy), d(Sx,fx), d(Ty,gy), d(Ty,fx)\} \\ + \beta[d(Ty,fx)+d(Sx,gy)] \end{aligned}$$

where $\alpha,\beta \in \mathbb{R}_+$ *such that* $\alpha+\beta<\frac{1}{2}$.

Proof. The proof of each contraction condition in this corollary follows easily from Theorem 2 in view of Examples 2-25.

Remark 5.
- *Here we point out a fallacies:*
  1. *(e.g. equation (6)) in the proof of Theorem 8 of Abbas et al. [3] wherein authors used* $\frac{z_1z_2}{1+z_2} \precsim z_1$ *which is not true in general (e.g. take* $z_1=1$ *and* $z_2=1+i$ *then* $\frac{1+i}{2+i} \not\precsim 1$*).*
  2. *(e.g in page 4 second line) in the proof of Theorem 8 of Abbas et al. [3] wherein authors used* $\psi(z_1) \precsim \psi(z_2) \Longrightarrow z_1 \precsim z_2$ *where* $\psi \in \Psi$ *which is not true in general (e.g. define* $\psi : \mathbb{C}_+ \longrightarrow \mathbb{C}_+$ *by* $\psi(z)=z^8+iz^4 \ \ \forall z \in \mathbb{C}_+$*, then* $\psi \in \Psi$ *and* $\psi(\frac{i}{2}) = \frac{1}{256}+\frac{i}{16} \precsim 1+i=\psi(1)$ *,* $\frac{i}{2} \not\precsim 1$ *)*

- *Contraction condition* (a_3) *generalizes and improves third case of Theorems 8 and 11 of [3]. In particular, substituting* $S=T$ *and* $f=g=I$*, we get third case of Theorem 8 of Abbas et al [3]. Also, setting* $S=T$ *and* $f=g$*, we get third case of Theorem 11 of Abbas et al [3].*

- *Contraction condition* (a_4) *(with* $\Delta = d(fx,gy)$*) generalizes and improves first case of Theorems 8 and 11 of [3]. In particular, substituting* $S=T$ *and* $f=g=I$*, we get first case of Theorem 8 of Abbas et al [3]. Also, setting* $S=T$ *and* $f=g$*, we get first case of Theorem 11 of Abbas et al [3].*

- (a_9) *represents a generalization form of Theorem 2.1 of [4]. In particular, substituting* $a_4=a_5=a_6=0$ *in* (a_9)*, we get Theorem 2.1 of Bhatt et al. [4].*

- (a_{14}) *generalizes Theorem 2.1 of [22]. In particular, substituting* $f=g=I$ *in* (a_{14})*, we get Theorem 2.1 of Verma et al. [22].*

- (a_{24}) *generalizes Corollary 2.2 of Verma et al. [22].*

Next, we have the following corollary which can be viewed as a generalization of Corollary 5.

Corollary 6. *The conclusions of Theorem (2) remain true if for all $x, y \in X$ the implicit relation (1) is replaced by one of the following:*

(b_1)

$$\begin{aligned} d(S^l x, T^m y) &\precsim \lambda_1(d(f^n x, g^s y)) d(f^n x, g^s y) \\ &+ \lambda_2(d(f^n x, g^s y)) \frac{d(T^m y, g^s y) d(S^l x, g^s y)}{1 + d(f^n x, g^s y)}, \end{aligned}$$

where $\lambda_1, \lambda_2 : \mathbb{C}_+ \longrightarrow [0,1)$ are given mappings.

(b_2)

$$\psi(d(S^l x, T^m y)) \precsim \psi\left(\frac{d(T^m y, g^s y) d(S^l x, g^s y)}{1 + d(f^n x, g^s y)}\right) + \phi\left(\frac{d(T^m y, g^s y) d(S^l x, g^s y)}{1 + d(f^n x, g^s y)}\right)$$

where $\psi \in \Psi$ and $\phi \in \Phi$.

(b_3)

$$\psi(d(S^l x, T^m y)) \precsim \psi\left(\frac{d(T^m y, g^s y) d(S^l x, g^s y)}{1 + d(f^n x, g^s y)}\right) - \phi\left(\frac{d(T^m y, g^s y) d(S^l x, g^s y)}{1 + d(f^n x, g^s y)}\right)$$

where $\psi \in \Psi$ and $\phi \in \Phi$.

(b_4)

$$\psi\big(d(S^l x, T^m y)\big) \precsim \psi(\Delta) - \phi(\Delta)$$

where $\Delta \in \left\{ d(f^n x, g^s y), d(S^l x, f^n x), \frac{d(S^l x, g^s y)}{1 + d(f^n x, g^s y)} \right\}$, $\psi \in \Psi$ with $\psi(z_1) \precsim \psi(z_2) \Longleftrightarrow z_1 \precsim z_2$ and $\phi \in \Phi$.

(b_5)

$$d(S^l x, T^m y) \precsim \begin{cases} \lambda d(f^n x, g^s y) + \mu \frac{d(S^l x, f^n x) d(T^m y, g^s y)}{\Delta} & \\ \qquad + \gamma \frac{d(T^m y, f^n x) d(S^l x, g^s y)}{\Delta}, & if \Delta \neq 0; \\ 0, & if \Delta = 0. \end{cases}$$

where $\Delta = d(T^m y, g^s y) + d(S^l x, g^s y)$ and $\lambda, \mu, \gamma \in \mathbb{R}_+$ such that $\lambda + \mu < 1$ and $\lambda + \gamma < 1$.

(b_6)

$$d(S^lx,T^my) \precsim \begin{cases} \lambda d(f^nx,g^sy)^{\frac{d(S^lx,f^nx)+d(T^my,g^sy)}{\Delta}} & \\ \qquad +\mu d(f^nx,g^sy)^{\frac{d(T^my,f^nx)+d(S^lx,g^sy)}{\Delta}}, & if\Delta \neq 0; \\ 0, & if\Delta = 0. \end{cases}$$

where $\Delta = d(f^nx,g^sy)+d(T^my,g^sy)$ *and* $\lambda,\mu \in \mathbb{R}_+$ *such that* $\lambda+2\mu<1$.

(b_7)

$$d(S^lx,T^my) \precsim \begin{cases} \alpha d(S^lx,f^nx)+\beta\frac{d(f^nx,g^sy)d(T^my,f^nx)}{\Delta} & \\ +\gamma\frac{d(T^my,g^sy)d(S^lx,g^sy)}{\Delta}+\delta\frac{d(f^nx,g^sy)d(S^lx,g^sy)}{\Delta} & \\ +\eta\frac{d(T^my,g^sy)d(T^my,f^nx)}{\Delta}, & if\Delta \neq 0; \\ 0, & if\Delta = 0. \end{cases}$$

where $\Delta = d(f^nx,g^sy)+d(T^my,g^sy)$ *and* $\alpha,\beta,\gamma,\delta,\eta \in \mathbb{R}_+$ *such that* $\alpha+\beta+\eta<1$ *and* $\beta+\delta<1$.

(b_8)

$$d(S^lx,T^my) \precsim \begin{cases} \alpha d(S^lx,f^nx) & \\ +\beta\frac{d(f^nx,g^sy)d(T^my,f^nx)}{\Delta}, & if\Delta \neq 0; \\ 0, & if\Delta = 0. \end{cases}$$

where $\Delta = d(S^lx,f^nx)+d(T^my,g^sy)+d(S^lx,g^sy)$ *and* $\alpha,\beta \in \mathbb{R}_+$ *such that* $\alpha+\beta<1$.

(b_9)

$$\begin{aligned} d(S^lx,T^my) &\precsim \alpha_1 d(f^nx,g^sy)+\alpha_2[d(S^lx,f^nx)+d(T^my,g^sy)] \\ &\quad +\alpha_3[d(T^my,f^nx)+d(S^lx,g^sy)]+\alpha_4\frac{d(T^my,g^sy)[1+d(S^lx,f^nx)]}{1+d(f^nx,g^sy)} \\ &\qquad +\alpha_5\frac{d(f^nx,g^sy)[1+d(S^lx,f^nx)+d(S^lx,g^sy)]}{1+d(f^nx,g^sy)}+\alpha_6 d(T^my,f^nx), \end{aligned}$$

where $\alpha_i \in \mathbb{R}_+, i=1,2,...,7,$ *such that* $\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5+2\alpha_6<1$.

(b_{10})

$$d(S^lx,T^my) \precsim \begin{cases} \lambda\frac{d(f^nx,g^sy)d(S^lx,g^sy)+d(T^my,g^sy)d(T^my,f^nx)}{\Delta}, & if\Delta \neq 0; \\ 0, & if\Delta = 0. \end{cases}$$

where $\Delta = d(f^nx,g^sy)+d(T^my,g^sy)+d(S^lx,g^sy)$ *and* $\lambda \in \mathbb{R}_+$ *such that* $\lambda<1$.

(b_{11})

$$d(S^lx,T^my)\precsim\begin{cases}\alpha_1 d(f^nx,g^sy)+\alpha_2\frac{d(S^lx,f^nx)d(T^my,g^sy)}{\Delta}\\+\alpha_3\frac{d(T^my,f^nx)d(f^nx,g^sy)}{\Delta}+\alpha_4\frac{d(S^lx,f^nx)d(f^nx,g^sy)}{\Delta}, & if\Delta\neq 0;\\0, & if\Delta=0.\end{cases}$$

where $\Delta = d(f^nx,g^sy)+d(S^lx,g^sy)$ *and* $\alpha_i\in\mathbb{R}_+, i=1,2,3,4$ *such that* $\alpha_1+\alpha_2+2\alpha_3+\alpha_4<1$.

(b_{12})

$$d(S^lx,T^my)\precsim\begin{cases}\alpha_1 d(S^lx,f^nx)+\alpha_2[d(f^nx,g^sy)+d(T^my,f^nx)]\\+\alpha_3[d(T^my,g^sy)+d(S^lx,g^sy)]\\+\alpha_4\frac{d(S^lx,f^nx)d(T^my,f^nx)+d(T^my,g^sy)d(S^lx,g^sy)}{\Delta}, & if\Delta\neq 0;\\0, & if\Delta=0.\end{cases}$$

where $\Delta = d(f^nx,g^sy)+d(T^my,g^sy)$ *and* $\alpha_i\in\mathbb{R}_+, i=1,2,3,4$ *such that* $\alpha_1+3\alpha_2+\alpha_3+\alpha_4<1$.

(b_{13})

$$\begin{aligned}d(S^lx,T^my)\precsim{}&\alpha d(S^lx,f^nx)+\beta\max\{d(f^nx,g^sy),d(S^lx,f^nx),d(S^lx,g^sy)\}\\&+\gamma\max\{d(S^lx,f^nx),d(T^my,f^nx)\},\end{aligned}$$

where $\alpha,\beta,\gamma\in\mathbb{R}_+$ *such that* $\alpha+\beta+2\gamma<1$.

(b_{14})

$$d(S^lx,T^my)\precsim\alpha\max\{d(f^nx,g^sy),d(S^lx,f^nx),d(T^my,g^sy),d(T^my,f^nx),d(S^lx,g^sy)\},$$

where $\alpha\in\mathbb{R}_+$ *such that* $\alpha<\frac{1}{2}$.

(b_{15})

$$\begin{aligned}\alpha_1 d(S^lx,T^my)\precsim{}&\alpha_2 d(f^nx,g^sy)+\alpha_3 d(S^lx,f^nx)+\alpha_4 d(T^my,g^sy)\\&+\alpha_5 d(T^my,f^nx)+\alpha_6 d(S^lx,g^sy),\end{aligned}$$

where $\alpha_i\in\mathbb{R}_+, i=1,2,...,6$ *such that* $\alpha_2+\alpha_3+\alpha_4+2\alpha_5+\alpha_6<\alpha_1$ *and* $\alpha_1>0$.

(b_{16})

$$d(S^lx,T^my)\precsim\alpha\max\{d(S^lx,f^nx)+d(T^my,g^sy),d(T^my,f^nx)+d(S^lx,g^sy)\},$$

where $\alpha\in\mathbb{R}_+$ *such that* $\alpha<\frac{1}{2}$.

(b_{17})

$$d(S^l x, T^m y) \precsim \alpha \max\{d(f^n x, g^s y), d(S^l x, f^n x), d(T^m y, g^s y), d(T^m y, f^n x) + d(S^l x, g^s y)\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{2}$.

(b_{18})

$$d(S^l x, T^m y) \precsim \alpha \max\{d(f^n x, g^s y), d(S^l x, f^n x), d(T^m y, f^n x), d(T^m y, g^s y) + d(S^l x, g^s y)\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{2}$.

(b_{19})

$$d(S^l x, T^m y) \precsim \alpha[d(T^m y, g^s y) + d(S^l x, g^s y)] + \alpha \max\{d(f^n x, g^s y), d(S^l x, f^n x), d(T^m y, f^n x)\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{3}$.

(b_{20})

$$d(S^l x, T^m y) \precsim \alpha \max\left\{\frac{2d(f^n x, g^s y) + d(T^m y, f^n x)}{2}, \frac{2d(f^n x, g^s y) + d(T^m y, g^s y)}{2}, \frac{2d(f^n x, g^s y) + d(S^l x, g^s y)}{2}\right\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{2}$.

(b_{21})

$$d(S^l x, T^m y) \precsim \alpha \max\left\{d(S^l x, f^n x), \frac{d(f^n x, g^s y) + d(T^m y, f^n x)}{2}, \frac{d(T^m y, g^s y) + d(S^l x, g^s y)}{2}\right\},$$

where $\alpha \in \mathbb{R}_+$ *such that* $\alpha < \frac{2}{3}$.

(b_{22})

$$d^2(S^l x, T^m y) \precsim \alpha d(S^l x, T^m y)\Big[d(f^n x, g^s y) + d(S^l x, f^n x) + d(T^m y, f^n x)\Big] + \beta d(T^m y, g^s y) d(S^l x, g^s y),$$

where $\alpha, \beta \in \mathbb{R}_+$ *such that* $\alpha < \frac{1}{4}$.

(b_{23})

$$d^2(S^lx,T^my) \precsim d(S^lx,T^my)\Big[\alpha_1 d(f^nx,g^sy)+\alpha_2 d(S^lx,f^nx)+\alpha_3 d(T^my,g^sy)\Big] + \alpha_4 d(T^my,f^nx)d(S^lx,g^sy),$$

where $\alpha_i \in \mathbb{R}_+, i = 1,2,3,4$ *such that* $\alpha_1+\alpha_2+\alpha_3 < 1$ *and* $\alpha_1+\alpha_4 < 1$.

(b_{24})

$$d(S^lx,T^my) \precsim \alpha \max\{d(f^nx,g^sy), d(S^lx,f^nx), d(T^my,g^sy), d(T^my,f^nx)\} + \beta[d(T^my,f^nx)+d(S^lx,g^sy)]$$

where $\alpha,\beta \in \mathbb{R}_+$ *such that* $\alpha+\beta < \frac{1}{2}$.

Proof. The proof of each contraction condition in this corollary follows easily from Corollary 4 in view of Examples 2-25.

4. APPLICATION TO HAMMERSTEIN INTEGRAL EQUATIONS AND URYSOHN INTEGRAL EQUATIONS

In this section, we show that contraction condition (a_1) of Corollary 5 can be applied to prove the existence and uniqueness of common solution for the system of Hammerstein integral equations:

$$x(t) = \psi_j(t) + \int_a^b k_j(t,s)f_j(s,x(s))ds, \tag{12}$$

where $t \in [a,b] \subseteq \mathbb{R}, x, \psi_j \in X, k_j : [a,b]\times[a,b] \longrightarrow \mathbb{R}^n$ and $f_j : [a,b]\times\mathbb{R}^n \longrightarrow \mathbb{R}^n, j = 1,2$.

Also, contraction condition (a_2) of Corollary 5 can be applied to prove the existence and uniqueness of common solution for the system of Urysohn integral equations:

$$x(t) = \varphi_j(t) + \int_a^b k_j(t,s,x(s))ds, \tag{13}$$

where $t \in [a,b] \subseteq \mathbb{R}, x, \varphi_j \in X, k_j : [a,b]\times[a,b]\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $j = 1,2$.

Throughout this section, we will use the following symbols:

$$\begin{aligned}
H_j\big(x(t)\big) &= \int_a^b k_j(t,s)f_j(s,x(s))ds,\ j=1,2,\\
U_j\big(x(t)\big) &= \int_a^b k_j(t,s,x(s))ds,\ j=1,2,\\
\Omega_{xy}(t) &= \|H_1x(t)+\psi_1(t)-H_2y(t)-\psi_2(t)\|_\infty\sqrt{1+a^2}\ e^{i\arctan a},\\
\Lambda_{xy}(t) &= \|x(t)-y(t)\|\sqrt{1+a^2}\ e^{i\arctan a},\\
\Xi_{xy}(t) &= \left(\frac{\|H_2y(t)+\psi_2(t)-y(t)\|_\infty\sqrt{1+a^2}\ e^{i\arctan a}}{1+\max_{t\in[a,b]}\Lambda_{xy}(t)}\right)\\
&\times\ \|H_1x(t)+\psi_1(t)-y(t)\|_\infty\sqrt{1+a^2}\ e^{i\arctan a},\\
\Omega^*_{xy}(t) &= \|U_1x(t)+\varphi_1(t)-U_2y(t)-\varphi_2(t)\|_\infty\sqrt{1+a^2}\ e^{i\arctan a},
\end{aligned}$$

$$\begin{aligned}
\Lambda^*_{xy}(t) = \left(\frac{\|U_2y(t)+\varphi_2(t)-y(t)\|_\infty\sqrt{1+a^2}\ e^{i\arctan a}}{1+\max_{t\in[a,b]}\Lambda_{xy}(t)}\right)\\
\times\|U_1x(t)+\varphi_1(t)-y(t)\|_\infty\sqrt{1+a^2}\ e^{i\arctan a},
\end{aligned}$$

Theorem 7. *Let $X=C\big([a,b],\mathbb{R}^n\big)$, $a>0$. Consider system (12) of Hammerstein integral equations. Suppose that k_1,k_2,f_1 and f_2 are such that $H_1(x),H_2(x)\in X$ for all $x\in X$. If there exist mappings $\lambda_1,\lambda_2:\mathbb{C}_+\longrightarrow[0,1)$ such that for each $x,y\in X$ and $t\in[0,1)$, we have*

$$\Omega_{xy}(t)\ \precsim\ \lambda_1\left(\max_{t\in[a,b]}\Lambda_{xy}(t)\right)\Lambda_{xy}(t)+\lambda_2\left(\max_{t\in[a,b]}\Lambda_{xy}(t)\right)\Xi_{xy}(t).$$

Then the system of integral equations (12) has a unique common solution.

Proof. Define a metric $d:X\times X\longrightarrow\mathbb{C}_+$ by

$$d(x,y)=\max_{t\in[a,b]}\|x(t)-y(t)\|\sqrt{1+a^2}\ e^{i\arctan a}.$$

Then (X,d) is a complete complex valued metric space. Define two mappings $S,T:X\longrightarrow X$ as follows:

$$S\big(x(t)\big)=\psi_1(t)+H_1\big(x(t)\big)=\psi_1(t)+\int_a^b k_1(t,s)f_1(s,x(s))ds,$$

$$T\big(x(t)\big) = \psi_2(t) + H_2\big(x(t)\big) = \psi_2(t) + \int_a^b k_2(t,s) f_2(s, x(s))ds.$$

Let $x, y \in X$. Then we have
$d(Sx, Ty) = \max_{t\in[a,b]} \|H_1x(t) + \psi_1(t) - H_2y(t) - \psi_2(t)\| \sqrt{1+a^2}\ e^{i \arctan a}$
$d(Ty, y) = \max_{t\in[a,b]} \|H_2y(t) + \psi_2(t) - y(t)\| \sqrt{1+a^2}\ e^{i \arctan a}$
$d(Sx, y) = \max_{t\in[a,b]} \|H_1x(t) + \psi_1(t) - y(t)\| \sqrt{1+a^2}\ e^{i \arctan a}$.

From assumption (14), for each $t \in [a,b]$ we have

$$\begin{aligned}\Omega_{xy}(t) &\precsim \lambda_1\left(\max_{t\in[a,b]} \Lambda_{xy}(t)\right)\Lambda_{xy}(t) + \lambda_2\left(\max_{t\in[a,b]} \Lambda_{xy}(t)\right)\Xi_{xy}(t)\\ &\precsim \lambda_1\left(\max_{t\in[a,b]} \Lambda_{xy}(t)\right)\max_{t\in[a,b]}\Lambda_{xy}(t) + \lambda_2\left(\max_{t\in[a,b]} \Lambda_{xy}(t)\right)\max_{t\in[a,b]}\Xi_{xy}(t).\end{aligned}$$

this implies that

$$\max_{t\in[a,b]}\Omega_{xy}(t) \precsim \lambda_1\left(\max_{t\in[a,b]} \Lambda_{xy}(t)\right)\max_{t\in[a,b]}\Lambda_{xy}(t) + \lambda_2\left(\max_{t\in[a,b]} \Lambda_{xy}(t)\right)\max_{t\in[a,b]}\Xi_{xy}(t).$$

That is,

$$d(Sx, Ty) \precsim \lambda_1(d(x,y))d(x,y) + \lambda_2(d(x,y))\frac{d(Ty,y)d(Sx,y)}{1+d(x,y)}$$

Now, we can apply contraction condition (a_1) in corollary (5) with $f = g = I$. Therefor we get that the Hammerstein integral equations in system (12) have a unique solution.

Theorem 8. *Let $X = C\big([a,b], \mathbb{R}^n\big)$, $a > 0$. Consider system (13) of Urysohn integral equations. Suppose that k_1 and k_2 are such that $U_1(x), U_2(x) \in X$ for all $x \in X$. If for each $x, y \in X$ and $t \in [a,b]$, we have*

$$\Omega^*_{xy}(t) \precsim \Lambda^*_{xy}(t) \tag{14}$$

Then the system of integral equations (13) has a unique common solution.

Proof. Let $X = C\big([a,b], \mathbb{R}^n\big)$, $a > 0$ and $d : X \times X \longrightarrow \mathbb{C}_+$ be defined by

$$d(x,y) = \max_{t\in[a,b]} \|x(t) - y(t)\| \sqrt{1+a^2}\ e^{i \arctan a}.$$

Then (X, d) is a complete complex valued metric space. Define two mappings $S, T : X \longrightarrow X$ as follows:

$$S\big(x(t)\big) = \varphi_1(t) + U_1\big(x(t)\big) = \varphi_1(t) + \int_a^b k_1(t, s, x(s))ds,$$

$$T\big(x(t)\big) = \varphi_2(t) + U_2\big(x(t)\big) = \varphi_2(t) + \int_a^b k_2(t,s,x(s))ds.$$

Let $x, y \in X$. Then we have
$d(Sx,Ty) = \max_{t\in[a,b]} \|U_1x(t) + \varphi_1(t) - U_2y(t) - \psi_2(t)\|\sqrt{1+a^2}\ e^{i\arctan a}$
$d(Sx,x) = \max_{t\in[a,b]} \|U_1x(t) + \varphi_1(t) - x(t)\|\sqrt{1+a^2}\ e^{i\arctan a}$
$d(Ty,y) = \max_{t\in[a,b]} \|U_2y(t) + \varphi_2(t) - y(t)\|\sqrt{1+a^2}\ e^{i\arctan a}$
$d(Sx,y) = \max_{t\in[a,b]} \|U_1x(t) + \varphi_1(t) - y(t)\|\sqrt{1+a^2}\ e^{i\arctan a}$.

On using assumption (14), for each $t \in [a,b]$ we have

$$\begin{aligned}\Omega^*_{xy}(t) &\precsim \Lambda^*_{xy}(t)\\ &\precsim \max_{t\in[a,b]} \Lambda^*_{xy}(t).\end{aligned}$$

this implies that

$$\max_{t\in[a,b]} \Omega^*_{xy}(t) \precsim \max_{t\in[a,b]} \Lambda^*_{xy}(t).$$

this yields that

$$\begin{aligned}\psi\big(\max_{t\in[a,b]} \Omega^*_{xy}(t)\big) &\precsim \psi\big(\max_{t\in[a,b]} \Lambda^*_{xy}(t)\big)\\ &\precsim \psi\big(\max_{t\in[a,b]} \Lambda^*_{xy}(t)\big) + \phi\big(\max_{t\in[a,b]} \Lambda^*_{xy}(t)\big).\end{aligned}$$

That is,

$$\psi(d(Sx,Ty)) \precsim \psi\left(\frac{d(Ty,y)d(Sx,y)}{1+d(x,y)}\right) + \phi\left(\frac{d(Ty,y)d(Sx,y)}{1+d(x,y)}\right)$$

Now, we can apply contraction condition (a_2) in corollary (5) with $f = g = I$. Therefor we get that the Urysohn integral equations in system (13) have a unique solution.

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Waleed Mohammad Alfaqih
Department of Mathematics,
Hajjah University,
Hajjah,
Yemen.
Email: *waleedmohd2016@gmail.com*