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CONCERNING THE ANALYSIS OF FRACTIONAL CALCULUS AND TRANSFORMS OF MULTIINDEX MITTAG-LEFFLER FUNCTION

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Abstract. The aim of this paper is to introduce a new extended multiindex Mittag-Leffler function by means of the extended beta function. We establish some interesting properties of extended multiindex Mittag-Leffler function, such as differential formulas, Mellin transform, Laplace transform, Euler-Beta transform, and Whittaker transform. Further, we evaluate some results assosciated with the Riemann-Liouville fractional integral and differential operators of the extended multiindex Mittag-Leffler function.

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1. Introduction and Preliminaries

Recently, many authors have introduced and further investigated many different fascinating expansions and results of various special functions (see [1]-[5], [7]-[15],[25],[26],[32]). Concerning his method of summation of some divergent series, Mittag-Leffler [17] introduced the Mittag-Leffler function. We have realized its significance over the past twenty years because of its application to physics, chemistry, biology, engineering, and applied sciences.

In 1903, Mittag -Leffler [17, 18] introduced the Mittag-Leffler function as,

$$\mathbf{E}_{\mu}(a) = \sum_{l=0}^{\infty} \frac{a^l}{\Gamma(\mu l + 1)} \ , \qquad (a, \mu \in \mathbb{C}; R(\mu) > 0).$$

In 1905, $E_{\mu,\tau}(a)$ a generalization of $E_{\mu}(a)$ was introduced by Wiman [29] in the form given as

$$\mathbf{E}_{\mu,\tau}(a)=\sum_{l=0}^{\infty}\frac{a^l}{\Gamma(\mu l+\tau)}\ ,\qquad (a,\mu,\tau\in\mathbb{C};R(\mu)>0,R(\tau)>0).$$

Recently, in 2009 Salim [23] introduced and investigated a new generalized Mittag-Leffler function

$$\mathbf{E}_{\mu,\tau}^{\lambda_1,\lambda_2}(a)=\sum_{l=0}^{\infty}\frac{(\lambda_1)_l\ a^l}{\Gamma(\mu l+\tau)\ (\lambda_2)_l}\ ,\tag{1}$$

$(\mu,\tau,a,\lambda_1,\lambda_2\in\mathbb{C};\ \min\{R(\mu),R(\tau),R(\lambda_2)\}>0;R(\lambda_2)>R(\lambda_1)>0)$,

where $(\lambda_1)_l$ is the classical pochhammer symbol [22], which is defined for $\lambda,l\in\mathbb{C}$ by means of the Gamma function Γ as

$$(\lambda_1)_l=\frac{\Gamma(\lambda_1+l)}{\Gamma(\lambda_1)}=\left\{\begin{array}{ll}1, & (l=0,\ \lambda_1\in\mathbb{C}\setminus\{0\}),\\ \lambda_1(\lambda_1+1)...(\lambda_1+n-1), & (l\in\mathbb{N},\lambda_1\in\mathbb{C})\end{array}\right\}.$$

Here, the classical Gamma function [22] is defined as

$$\Gamma(p)=\int_0^{\infty}e^{-t}\,t^{p-1}\,dt,\qquad (R(p)>0).$$

In this paper, we introduce multiindex Mittag-Leffler function, for $n>1$ as

$$\mathbf{E}_{(\mu_i,\tau_i)_1^n}^{\lambda_1,\lambda_2}(a)=\sum_{l=0}^{\infty}\frac{(\lambda_1)_l\,a^l}{\prod_{i=1}^{n}\Gamma(\mu_i l+\tau_i)\,(\lambda_2)_l},\tag{2}$$

$(a,\mu_i,\tau_i,\lambda_1,\lambda_2\in\mathbb{C};\ \min\{R(\mu_i),R(\tau_i),R(\lambda_2)\}>0;R(\lambda_2)>R(\lambda_1)>0;\ i=1,...,n)$.

Which for $i=1$ will reduce to $E_{\mu_1,\tau_1}^{\lambda_1,\lambda_2}(a)$ given in (1).

For an appropriately-bounded sequence $\{j_m\}_{m\in\mathbb{N}_0}$ of arbitrary (real or complex) numbers, the function $\Omega\left(\{j_m\}_{m\in\mathbb{N}_0};\ \xi\right)$ is defined by Srivastava et al. [27] in 2012 as,

$$\Omega\left(\{j_m\}_{m\in\mathbb{N}_0};\ \xi\right) := \left\{ \begin{array}{ll} \sum_{l=0}^{\infty}\ j_l\ \frac{\xi^l}{l!} & (\mid \xi \mid < R;\ 0<R<\infty;\ j_0 := 1) \\ \\ v_0\ \xi^{\mu}\ \exp(\xi)\left[1+O\left(\frac{1}{\xi}\right)\right] & (\Re(\xi)\to\infty : v_0>0;\ \mu\in\mathbb{C}) \end{array} \right\}, \tag{3}$$

for some suitable constant v_0 and μ essentially dependent on the sequence $\{j_m\}_{m\in\mathbb{N}_0}$. In terms of (3), they introduced the following extensions of the gamma function, beta function, and the Gauss hypergeometric function

$$\Gamma_x^{(\{j_m\}_{m\in\mathbb{N}_0})}(\xi_1) = \int_0^{\infty} {\xi_2}^{\xi_1-1}\,\Omega\left(\{j_m\}_{m\in\mathbb{N}_0};\ -\xi_2-\frac{x}{\xi_2}\right)d\xi_2, \tag{4}$$

$$(\Re(\xi_1)>0; \Re(x)\geq 0)$$

$$B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\mu,\tau;x) = \int_0^1 {\xi_2}^{\mu-1}(1-\xi_2)^{\tau-1}\Omega\left(\{j_m\}_{m\in\mathbb{N}_0};\ -\frac{x}{\xi_2(1-\xi_2)}\right)d\xi_2, \tag{5}$$

$$(\min\{\Re(\mu),\Re(\tau)\}>0; \Re(x)\geq 0)$$

and

$$F_x^{(\{j_m\}_{m\in\mathbb{N}_0})}(b,\lambda_1;\lambda_2;\ \xi_1) = \sum_{l=0}^{\infty}(b)_l\ \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\frac{\xi_1^l}{l!}\ , \tag{6}$$

$$(\ \Re(x)\geq 0;\ \mid \xi_1\mid<1;\ \Re(\lambda_2)>\Re(\lambda_1)>0)$$

respectively.

For different special choices of the sequence $\{j_m\}_{m\in\mathbb{N}_0}$, eqs. (4)-(6) would reduce to some different extensions of Gamma, Beta and hypergeometric functions. In prticular, if we put

$$j_m = \frac{(\rho_1)_m}{(\rho_2)_m} \qquad (m\in\mathbb{N}_0),$$

equations (4)-(6) become the following extensions by Özergin et al. [20]

$$\Gamma_x^{(\rho_1,\rho_2)}(\xi_1) = \int_0^{\infty} {\xi_2}^{\xi_1-1}\ {}_1F_1\left(\rho_1;\rho_2;\ -\xi_2-\frac{x}{\xi_2}\right)d\xi_2,$$

$$(\Re(x)\geq 0;\ \min\{\Re(\xi_1),\Re(\rho_1),\Re(\rho_2)\}>0).$$

$$B^{(\rho_1,\rho_2)}(\mu,\tau;x)=\int_0^1 {\xi_2}^{\mu-1}(1-\xi_2)^{\tau-1}\ {}_1F_1\left(\rho_1;\rho_2;-\frac{b}{\xi_2(1-\xi_2)}\right)d\xi_2,$$

$$(R(x)\geq 0;\ \min\{R(\mu),R(\tau),R(\rho_1),R(\rho_2)\}>0).$$

and

$$F_x^{(\rho_1,\rho_2)}(b,\lambda_1;\lambda_2;\xi_1)=\frac{1}{B(\lambda_1,\lambda_2-\lambda_1)}\sum_{l=0}^{\infty}(b)_l\,B^{(\rho_1,\rho_2)}(\lambda_1+l,\lambda_2-\lambda_1;x)\frac{{\xi_1}^l}{l!}\ ,$$

$$(|\ \xi_1\ |<1;\ \min\{R(\rho_1),R(\rho_2)\}>0;R(\lambda_2)>R(\lambda_1)>0;\ R(x)\geq 0)$$

respectively.

Take $j_m=1$, and we can obtain the functions studied by Chaudhry et al. [3, 4] from eqs. (4)-(6) as follows

$$\Gamma_x(\xi_1)=\int_0^\infty {\xi_2}^{\xi_1-1}\exp\left(-\xi_2-\frac{x}{\xi_2}\right)d\xi_2,\qquad (R(x)>0;\ \xi_1\in\mathbb{C}),$$

$$B(\mu,\tau;x)=\int_0^1 {\xi_2}^{\mu-1}(1-\xi_2)^{\tau-1}\exp\left(-\frac{x}{\xi_2(1-\xi_2)}\right)d\xi_2,\qquad (R(x)>0),$$

and

$$F_x(b,\lambda_1;\lambda_2;\xi_1)=\sum_{l=0}^{\infty}(b)_l\frac{B(\lambda_1+l,\lambda_2-\lambda_1;\ x)}{B(\lambda_1,\lambda_2-\lambda_1)}\frac{{\xi_1}^l}{l!}\ ,$$

$$(R(\lambda_2)>R(\lambda_1)>0;\ x\geq 0,\ |\ \xi_1\ |<1)$$

respectively.

When $j_m=0$ ($m\in\mathbb{N}_0$) or for $x=0$, eqs. (4)-(6) reduce to their classical forms [16, 19].

The generalized Wright hypergeometric function ${}_g\Psi_h[t]$, also known as the Fox-Wright function [30, 31] defined as

$$ {}_g\Psi_h[t]={}_g\Psi_h\left[\begin{array}{c}(\alpha_1,\acute{\alpha}_1),...,(\alpha_g,\acute{\alpha}_g);\\(\beta_1,\acute{\beta}_1),...,(\beta_h,\acute{\beta}_h);\end{array}\ t\right] \tag{7}$$

$$= \sum_{l=0}^{\infty} \frac{\Gamma(\alpha_1 + \acute{\alpha}_1 l)...\Gamma(\alpha_g + \acute{\alpha}_g l)}{\Gamma(\beta_1 + \acute{\beta}_1 l)...\Gamma(\beta_h + \acute{\beta}_h l)} \frac{t^l}{l!} \tag{8}$$

$$= H^{1,g}_{g,h+1}\left[-t \left| \begin{array}{c} (1-\alpha_1, \acute{\alpha}_1), ..., (1-\alpha_g, \acute{\alpha}_g) \\ (0,1), (1-\beta_1, \acute{\beta}_1), ..., (1-\beta_h, \acute{\beta}_h) \end{array} \right.\right],$$

here, $H^{i,k}_{g,h}[t]$ denotes the Fox-H function [6] defined by a Mellin-Barnes integral as,

$$H^{i,k}_{g,h}\left[t \left| \begin{array}{c} (\alpha_1, \acute{\alpha}_1), (\alpha_2, \acute{\alpha}_2), ..., (\alpha_g, \acute{\alpha}_g) \\ (\beta_1, \acute{\beta}_1), (\beta_2, \acute{\beta}_2), ..., (\beta_h, \acute{\beta}_h) \end{array} \right.\right]$$

$$= \frac{1}{2\pi i} \int_L \frac{\prod_{n=1}^{i} \Gamma(\beta_n + \acute{\beta}_n \xi_1) \prod_{n=1}^{k} \Gamma(1-\alpha_n + \acute{\alpha}_n \xi_1)}{\prod_{n=i+1}^{h} \Gamma(1-\beta_n + \acute{\beta}_n \xi_1) \prod_{n=k+1}^{g} \Gamma(\alpha_n + \acute{\alpha}_n \xi_1)} t^{-\xi_1} d\xi_1, \tag{9}$$

where L is a contour separating the poles of the two factors in the numerator.

For $\acute{\alpha}_1 = ... = \acute{\alpha}_g = 1$, $\acute{\beta}_1 = ... = \acute{\beta}_h = 1$ eq. (9) reduces to the generalized hypergeometric function ${}_gF_h$ [32]

$${}_g\Psi_h\left[\begin{array}{c} (\alpha_1, 1), ..., (\alpha_g, 1); \\ (\beta_1, 1), ..., (\beta_h, 1); \end{array} t\right] = \frac{\Gamma(\alpha_1)...\Gamma(\alpha_g)}{\Gamma(\beta_1)...\Gamma(\beta_h)} \, {}_gF_h(\alpha_1, ..., \alpha_g; \beta_1, ..., \beta_h; t).$$

We extend the multiindex Mittag-Leffler function given in (2) by means of the extended beta function $B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\mu, \tau; x)$ defined by (5), mainly driven by the significant applications of these extended hypergeometric functions and discuss some properties such as differential formulas, and some integral transform namely Euler-Beta transform, Laplace transform, Mellin transform, and Whittaker transform. Further, we investigate some results involving Riemann Liouville fractional integrals and derivatives of the extended multiindex Mittag-Leffler function.

2. Extended Multiindex Mittag-Leffler Function

Using (5) with the approach

$$\frac{(\lambda_1)_l}{(\lambda_2)_l} = \frac{B(\lambda_1 + l, \lambda_2 - \lambda_1)}{B(\lambda_1, \lambda_2 - \lambda_1)} \to \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1 + l, \lambda_2 - \lambda_1; x)}{B(\lambda_1, \lambda_2 - \lambda_1)} \ ,$$

in (2), we define the new multiindex Mittag-Leffler function as

$$\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(a;x) = \sum_{l=0}^{\infty} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1 + l, \lambda_2 - \lambda_1; x)}{B(\lambda_1, \lambda_2 - \lambda_1)} \ \frac{a^l}{\prod\limits_{i=1}^{n} \Gamma(\mu_i l + \tau_i)} \ , \tag{10}$$

($\mu_i, \tau_i, a, \lambda_1, \lambda_2 \in \mathbb{C},$ *and* $x \geq 0;\ \min\{R(\mu_i), R(\tau_i), R(\lambda_2)\} > 0; R(\lambda_2) > R(\lambda_1) > 0,\ i = 1, ..., n$).

Remark 1. *If we put* $j_m = \frac{(\rho_1)_m}{(\rho_2)_m}$ $(m \in \mathbb{N}_0)$*, which gives another case of* (10)

$$\mathbf{E}^{(\rho_1,\rho_2);\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(a;x) = \sum_{l=0}^{\infty} \frac{B^{(\rho_1,\rho_2)}(\lambda_1 + l, \lambda_2 - \lambda_1; x)}{B(\lambda_1, \lambda_2 - \lambda_1)} \ \frac{a^l}{\prod\limits_{i=1}^{n} \Gamma(\mu_i l + \tau_i)} \tag{11}$$

($\mu_i, \tau_i, a, \lambda_1, \lambda_2 \in \mathbb{C},$ *and* $x \geq 0;\ \min\{R(\mu_i), R(\tau_i), R(\lambda_2)\} > 0; R(\lambda_2) > R(\lambda_1) > 0,\ i = 1, ..., n$).

For $x = 0$ *or* $j_m = 0$ $(m \in \mathbb{N}_0)$*, eq.* (10) *becomes* (2).

Remark 2. *Put* $\mu_i = 0,\ \tau_i = 1$ $(i = 1, ..., n)$ *in* (10) *and* (11) *we get special cases of multiindex Mittag-Leffler function*

$$\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(0_i,1_i)_1^n}(a;x) = \frac{\Gamma(\lambda_2)}{\Gamma(\lambda_1)} {}_2\Psi_1^{(\{j_m\}_{m\in\mathbb{N}_0})}\left[\begin{matrix}(\lambda_1,1),(1,1); \\ (\lambda_2,1);\end{matrix} (a;x)\right],$$

and

$$\mathbf{E}^{(\rho_1,\rho_2);\lambda_1,\lambda_2}_{(0_i,1_i)_1^n}(a;x) = \frac{\Gamma(\lambda_2)}{\Gamma(\lambda_1)} {}_2\Psi_1^{(\rho_1,\rho_2)}\left[\begin{matrix}(\lambda_1,1),(1,1); \\ (\lambda_2,1);\end{matrix} (a;x)\right].$$

Moreover, for $j_m = 0,\ x = 0,\ \mu_i = 1$ $(i = 1, ..., n)$ *and* a *is replaced by* $-a$*, yields*

$$\mathbf{E}^{\lambda_1,\lambda_2}_{(1_i,\tau_i)_1^n}(-a) = \frac{\Gamma(\lambda_2)}{\Gamma(\lambda_1)} {}_2\Psi_{n+1}\left[\begin{matrix}(\lambda_1,1),(1,1); \\ (\lambda_2,1),(\tau_i,1)_1^n;\end{matrix} -a\right].$$

3. Derivative Properties of Extended Multiindex Mittag-Leffler Function

Here, we discuss some derivative properties of the extended multiindex Mittag-Leffler function (10).

Theorem 1. *The differential formula for the extended multiindex Mittag-Leffler function is as follows*

$$\left(\frac{d}{da}\right)^r \left[a^{\tau-1}\mathbf{E}_{(\mu_i,\tau_i)_1^n}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}(\eta a^{\mu};x)\right]$$
$$= a^{\tau-r-1}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\,{}_2\Psi_{n+1}\left[\begin{matrix}(\tau,\mu);(1,1);\\(\tau-r,\mu),(\tau_i,\mu_i)_1^n;\end{matrix}\ \eta a^{\mu}\right], \tag{12}$$

$(\mu_i,\tau_i,a,\lambda_1,\lambda_2\in\mathbb{C};\ R(\tau-r)>0, r\in\mathbb{N};\ R(\lambda_2)>R(\lambda_1)>0,\ i=1,...,n).$

Proof. Using (10) and employing term-wise r times differentiations on (12), yields

$$\left(\frac{d}{da}\right)^r \left[a^{\tau-1}\mathbf{E}_{(\mu_i,\tau_i)_1^n}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}(\eta a^{\mu};x)\right]$$
$$= \sum_{l=0}^{\infty}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\ \frac{\eta^l}{\prod_{i=1}^{n}\Gamma(\mu_i l+\tau_i)}\left(\frac{d}{da}\right)^r\left(a^{\mu l+\tau-1}\right) \tag{13}$$

$$= \sum_{l=0}^{\infty}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\ \frac{\Gamma(\mu\, l+\tau)\,\eta^l\,a^{\mu l+\tau-r-1}}{\Gamma(\mu l+\tau-r)\prod_{i=1}^{n}\Gamma(\mu_i l+\tau_i)}$$

$$= a^{\tau-r-1}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\ \sum_{l=0}^{\infty}\frac{\Gamma(\mu\, l+\tau)\,\Gamma(l+1)\,(\eta a^{\mu})^l}{\Gamma(\mu l+\tau-r)\,l!\,\prod_{i=1}^{n}\Gamma(\mu_i l+\tau_i)} \tag{14}$$

Using eq. (7) and (8) in (14), we get the required result.

Theorem 2. *We have the following derivative formula for the extended multiindex Mittag-Leffler function*

$$\left(\frac{d^r}{da^r}\right)\left[\mathbf{E}_{(\mu_i,\tau_i)_1^n}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}(a;x)\right] = \frac{(\lambda_1)_r}{(\lambda_2)_r}\sum_{l=0}^{\infty}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l+r,\lambda_2-\lambda_1;x)}{B(\lambda_1+r,\lambda_2-\lambda_1)}$$

$$\times \frac{(l+1)_r a^l}{\prod_{i=1}^{n} \Gamma(\mu_i l + \tau_i + \mu_i r)}, \tag{15}$$

$(\mu_i, \tau_i, \lambda_1, \lambda_2 \in \mathbb{C},\ and\ r \in \mathbb{N};\ R(x) > 0, R(\lambda_2) > R(\lambda_1) > 0,\ i = 1, ..., n)$.

Proof. Using (10) and applying differentiation with respect to a yields,

$$\left(\frac{d}{da}\right)\left[\mathbf{E}_{(\mu_i,\tau_i)_1^n}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}(a;x)\right]$$

$$= \frac{d}{da}\left[\sum_{l=0}^{\infty} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1 + l, \lambda_2 - \lambda_1; x)}{B(\lambda_1, \lambda_2 - \lambda_1)} \frac{a^l}{\prod_{i=1}^{n} \Gamma(\mu_i l + \tau_i)}\right]$$

$$= \sum_{l=1}^{\infty} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1 + l, \lambda_2 - \lambda_1; x)}{B(\lambda_1, \lambda_2 - \lambda_1)} \frac{a^{l-1} l}{\prod_{i=1}^{n} \Gamma(\mu_i l + \tau_i)}.$$

Replacing $l \to l + 1$ and using the result ([21], p.46)

$$B(\lambda_1, \lambda_2 - \lambda_1) = \frac{\lambda_2}{\lambda_1} B(\lambda_1 + 1, \lambda_2 - \lambda_1),$$

yields,

$$= \frac{\lambda_1}{\lambda_2} \sum_{l=0}^{\infty} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1 + l + 1, \lambda_2 - \lambda_1; x)}{B(\lambda_1 + 1, \lambda_2 - \lambda_1)} \frac{(l+1)a^l}{\prod_{i=1}^{n} \Gamma(\mu_i l + \tau_i + \mu_i)}.$$

Reapplying this term-wise differentiation rth time, we get

$$\left(\frac{d^r}{da^r}\right)\left[\mathbf{E}_{(\mu_i,\tau_i)_1^n}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}(a;x)\right] = \frac{(\lambda_1)_r}{(\lambda_2)_r} \sum_{l=0}^{\infty} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1 + l + r, \lambda_2 - \lambda_1; x)}{B(\lambda_1 + r, \lambda_2 - \lambda_1)}$$

$$\times \frac{(l+1)_r\, a^l}{\prod_{i=1}^{n} \Gamma(\mu_i l + \tau_i + \mu_i r)}.$$

Remark 3. *For $x = 0$ or $j_m = 0$ and $i = 1$, (15) becomes a known result [23]*

$$\left(\frac{d^r}{da^r}\right)\left[\mathbf{E}^{\lambda_1,\lambda_2}_{\mu_1,\tau_1}(a)\right] = \frac{(\lambda_1)_r}{(\lambda_2)_r}\sum_{l=0}^{\infty}\frac{(\lambda_1+r)_l}{(\lambda_2+r)_l}\,\frac{(l+1)_r\,a^l}{\Gamma(\mu_1 l+\tau_1+\mu_1 r)}.$$

4. Integral Transform of Extended Multiindex Mittag-Leffler function

In this section, we obtain the Euler-Beta transform, Laplace transform, Melli transform and the Whittaker transform for the extended multiindex Mittag-Leffler function.

1. Euler-Beta Transform

For the function $f(a)$, the Euler-Beta transform [30] is defined as

$$B\{f(a);p,q\} = \int_0^1 a^{p-1}(1-a)^{q-1}f(a)\,da. \tag{16}$$

Theorem 3. *The extended multiindex Mittag-Leffler function satisfies the following Euler-Beta transform*

$$\begin{aligned}&B\left\{\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(ba^{\rho_2};x);p,q\right\}\\ &=\Gamma(q)\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\ {}_2\Psi_{n+1}\left[\begin{matrix}(p,\rho_2),(1,1);\\(\tau_i,\mu_i)_1^n,(p+q,\rho_2);\end{matrix}\ b\right],\end{aligned} \tag{17}$$

$(\mu_i,\tau_i,\lambda_1,\lambda_2 \in \mathbb{C};\ R(x) > 0, R(p) > 0, R(q) > 0, R(\lambda_2) > R(\lambda_1) > 0,\ i = 1,...,n)$.

Proof. Applying (16) in (10), we get

$$\begin{aligned}&B\left\{\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(ba^{\rho_2};x);p,q\right\}\\ &=\int_0^1 a^{p-1}(1-a)^{q-1}\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(ba^{\rho_2};x)\,da\end{aligned}$$

$$= \int_0^1 a^{p-1}(1-a)^{q-1} \left(\sum_{l=0}^{\infty} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)} \; \frac{b^l a^{\rho_2 l}}{\prod_{i=1}^{n}\Gamma(\mu_i l+\tau_i)} \right) da$$

$$= \sum_{l=0}^{\infty} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)} \; \frac{b^l}{\prod_{i=1}^{n}\Gamma(\mu_i l+\tau_i)} \int_0^1 a^{p+\rho_2 l-1}(1-a)^{q-1}\,da$$

$$= \sum_{l=0}^{\infty} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)} \; \frac{b^l}{\prod_{i=1}^{n}\Gamma(\mu_i l+\tau_i)} \; \frac{\Gamma(p+\rho_2 l)\Gamma(q)}{\Gamma(p+q+\rho_2 l)}$$

$$= \Gamma(q)\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)} \sum_{l=0}^{\infty} \frac{\Gamma(p+\rho_2 l)\Gamma(l+1)\,b^l}{\prod_{i=1}^{n}\Gamma(\mu_i l+\tau_i)\Gamma(p+q+\rho_2 l)\,l!} \tag{18}$$

Using eq (7) and (8) in (18), we reach the result (17).

Corollary 4. *Substitute a by $(1-a)$ in $\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(a;x)$, yields*

$$B\left\{\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(b(1-a)^{\rho_2};x);p,q\right\}$$
$$= \int_0^1 a^{p-1}(1-a)^{q-1}\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(b(1-a)^{\rho_2};x)\,da$$
$$= \Gamma(p)\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)} \; {}_2\Psi_{n+1}\left[\begin{matrix}(q,\rho_2),(1,1); \\ (\tau_i,\mu_i)_1^n,(p+q,\rho_2);\end{matrix}\; b\right].$$

Remark 4. *For $x=0$ or $j_m=0$ and $i=1$ in (17), we obtain the known Euler-Beta transform for $\mathbf{E}^{\lambda_1,\lambda_2}_{\mu_1,\tau_1}(a)$ [23].*

2. Laplace Transform

The Laplace transform [24] of the function $f(a)$ is defined as:

$$\mathbf{L}\{f(a)\} = \int_0^{\infty} e^{-sa}f(a)\,da.$$

Theorem 5. *The extended multiindex Mittag-Leffler function satisfies the following Laplace transform*

$$\int_0^\infty a^{p-1}e^{-sa}\ \mathbf{E}_{(\mu_i,\tau_i)_1^n}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}(ba^{\rho_2};x)\,da$$
$$=\frac{1}{s^p}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\ {}_2\Psi_n\left[\begin{matrix}(p,\rho_2),(1,1); & \dfrac{b}{s^{\rho_2}}\\ (\tau_i,\mu_i)_1^n; & \end{matrix}\right], \tag{19}$$

$(\mu_i,\tau_i,\lambda_2,\lambda_1\in\mathbb{C};\ \ R(s)>0,R(x)>0,R(\lambda_2)>R(\lambda_1)>0,\ i=1,...,n)$.

Proof. Suppose the R.H.S. of (19) as I_1 and using (10), yields

$$I_1=\int_0^\infty a^{p-1}e^{-sa}\ \mathbf{E}_{(\mu_i,\tau_i)_1^n}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}(ba^{\rho_2};x)\,da$$
$$=\int_0^\infty a^{p-1}e^{-sa}\ \sum_{l=0}^\infty\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\ \frac{b^la^{\rho_2 l}}{\prod_{i=1}^n\Gamma(\mu_il+\tau_i)}\,da$$
$$=\sum_{l=0}^\infty\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\ \frac{b^l}{\prod_{i=1}^n\Gamma(\mu_il+\tau_i)}\ \int_0^\infty a^{p+\rho_2l-1}e^{-sa}\,da$$
$$=\frac{1}{s^p}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\ \sum_{l=0}^\infty\frac{\Gamma(p+\rho_2l)\Gamma(l+1)}{\prod_{i=1}^n\Gamma(\mu_il+\tau_i)\,l!}\left(\frac{b}{s^{\rho_2}}\right)^l \tag{20}$$

Now, using eq (7) and (8) in (20), we reach the result (19).

Remark 5. *For $x=0$ or $j_m=0$ and $i=1$ in* (19)*, gives the following Laplace transform for $\mathbf{E}_{\mu_1,\tau_1}^{\lambda_1,\lambda_2}(a)$ [23] as*

$$\int_0^\infty a^{p-1}e^{-sa}\ \mathbf{E}_{\mu_1,\tau_1}^{\lambda_1,\lambda_2}(ba^{\rho_2})\,da$$
$$=\frac{1}{s^p}\frac{\Gamma(\lambda_2)}{\Gamma(\lambda_1)}\ {}_3\Psi_2\left[\begin{matrix}(\lambda_1,1),(p,\rho_2),(1,1); & \dfrac{t}{s^{\rho_2}}\\ (\lambda_2,1),(\tau_1,\mu_1); & \end{matrix}\right].$$

3. Mellin Transform

The Mellin transform [27] of an appropriately-integrable function $f(a)$ with index s is defined as

$$M\{f(a); a \to s\} := \int_0^\infty a^{s-1} f(a)\, da \tag{21}$$

whenever the improper integral in (21) exists.

Theorem 6. *The extended multiindex Mittag-Leffler function satisfies the following Mellin transform*

$$M\{\mathbf{E}_{(\mu_i,\tau_i)_1^n}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}(a;x); x \to s\}$$

$$= \frac{\Gamma_0^{(\{j_m\})_{m\in\mathbb{N}_0})}(s)\Gamma(\lambda_2)\Gamma(\lambda_2-\lambda_1+s)}{\Gamma(\lambda_1)\Gamma(\lambda_2-\lambda_1)}\ {}_2\Psi_{n+1}\left[\begin{matrix}(\lambda_1+s,1),(1,1); \\ (\lambda_2+2s,1),(\tau_i,\mu_i)_1^n;\end{matrix}\ a\right], \tag{22}$$

$(R(s) > 0,$ *and* $R(\lambda_2-\lambda_1+s) > 0),$

where $\Gamma_0^{(\{j_m\})_{m\in\mathbb{N}_0})}$ *is a particular case of* (4) *for* $x = 0$.

Proof. Applying (21) in (10), yields

$$M\{\mathbf{E}_{(\mu_i,\tau_i)_1^n}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}(a;x); x \to s\}$$

$$= \int_0^\infty x^{s-1}\, \mathbf{E}_{(\mu_i,\tau_i)_1^n}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}(a;x)\, dx$$

$$= \int_0^\infty x^{s-1} \left(\sum_{l=0}^\infty \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)} \frac{a^l}{\prod_{i=1}^n \Gamma(\mu_i l+\tau_i)}\right) dx,$$

which, on further solving, gives,

$$= \frac{1}{B(\lambda_1,\lambda_2-\lambda_1)} \sum_{l=0}^\infty \frac{a^l}{\prod_{i=1}^n \Gamma(\mu_i l+\tau_i)} \int_0^\infty x^{s-1} B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)\, dx. \tag{23}$$

Using the known formula [27]

$$\int_0^\infty x^{s-1}B^{(\{j_m\}_{m\in\mathbb{N}_0})}(b,a;x)dx=\Gamma_0^{(\{j_m\}_{m\in\mathbb{N}_0})}(s)\ B(b+s,a+s),\quad R(s)>0,$$

in (23) we obtain,

$$\begin{aligned}&M\{\mathbf{E}_{(\mu_i,\tau_i)_1^n}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}(a;x);x\to s\}\\&=\frac{\Gamma_0^{(\{j_m\}_{m\in\mathbb{N}_0})}(s)}{B(\lambda_1,\lambda_2-\lambda_1)}\sum_{l=0}^{\infty}B(\lambda_1+l+s,\lambda_2-\lambda_1+s)\,\frac{a^l}{\prod_{i=1}^{n}\Gamma(\mu_i l+\tau_i)}\\&=\frac{\Gamma_0^{(\{j_m\}_{m\in\mathbb{N}_0})}(s)\,\Gamma(\lambda_2)\,\Gamma(\lambda_2-\lambda_1+s)}{\Gamma(\lambda_1)\,\Gamma(\lambda_2-\lambda_1)}\sum_{l=0}^{\infty}\frac{\Gamma(\lambda_1+s+l)\,\Gamma(l+1)\,a^l}{\Gamma(\lambda_2+l+2s)\prod_{i=1}^{n}\Gamma(\mu_i l+\tau_i)\,l!}\end{aligned}\tag{24}$$

Applying, (7) and (8) in (24), we reach the result (22).

4. Whittaker Transform

Theorem 7. *The extended multiindex Mittag-Leffler function satisfies the following Whittaker transform*

$$\begin{aligned}&\int_0^\infty a^{u-1}e^{\frac{-\nu a}{2}}\,W_{\alpha,\beta}(\nu a)\,\mathbf{E}_{(\mu_i,\tau_i)_1^n}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}(ba^{\rho_2};x)\,da\\&=\nu^{-u}\,\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\,{}_3\Psi_{n+1}\left[\begin{matrix}(\frac{1}{2}+\beta+u,\rho_2),(\frac{1}{2}-\beta+u,\rho_2),(1,1);\\(\tau_i,\mu_i)_1^n,(1-\alpha+u,\rho_2);\end{matrix}\ \frac{b}{\nu^{\rho_2}}\right].\end{aligned}\tag{25}$$

Proof. Put $\nu a=c$ in left hand side of (25), yields

$$\int_0^\infty \left(\frac{c}{\nu}\right)^{u-1} e^{\frac{-c}{2}} W_{\alpha,\beta}(c) \sum_{l=0}^{\infty} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)\, b^l}{B(\lambda_1,\lambda_2-\lambda_1)\prod_{i=1}^{n}\Gamma(\mu_i l+\tau_i)} \left(\frac{c}{\nu}\right)^{\rho_2 l} \frac{1}{\nu}\, dc$$

$$= \nu^{-u} \sum_{l=0}^{\infty} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)\, b^l}{B(\lambda_1,\lambda_2-\lambda_1)\prod_{i=1}^{n}\Gamma(\mu_i l+\tau_i)\, \nu^{\rho_2 l}} \int_0^\infty c^{\rho_2 l+u-1} e^{\frac{-c}{2}} W_{\alpha,\beta}(c)\, dc \tag{26}$$

Now, employing the formula

$$\int_0^\infty x^{u-1} e^{\frac{-x}{2}} W_{\alpha,\beta}(x)\, dx = \frac{\Gamma(\frac{1}{2}+\beta+u)\ \Gamma(\frac{1}{2}-\beta+u)}{\Gamma(1-\alpha+u)}, \ \left(R(u\pm\beta) > \frac{-1}{2}\right),$$

where, $W_{\alpha,\beta}(x)$ is whittaker function [28],
(26) becomes

$$= \nu^{-u} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}$$
$$\times \sum_{l=0}^{\infty} \frac{\Gamma(\frac{1}{2}+\beta+u+\rho_2 l)\ \Gamma(\frac{1}{2}-\beta+u+\rho_2 l)\ \Gamma(l+1)}{\Gamma(1-\alpha+u+\rho_2 l)\ \prod_{i=1}^{n}\Gamma(\mu_i l+\tau_i)\ l!} \left(\frac{b}{\nu^{\rho_2}}\right)^l \tag{27}$$

Using (7) and (8) in (27), we reach the result (25).

5. Fractional Property of Extended Multiindex Mittag-Leffler function

Here, we investigate some results associated with the right-sided Riemann-Liouville fractional integral operator I^α_{w+} and the derivative operator D^α_{w+}, which are defined respectively as follows [15, 24]:

$$(I^\alpha_{w+}\theta)(t) = \frac{1}{\Gamma(\alpha)} \int_w^t \frac{\theta(a)}{(t-a)^{1-\alpha}} da, \quad (R(\alpha) > 0,\ \alpha\in\mathbb{C}), \tag{28}$$

and

$$(D^{\alpha}_{w+}\theta)(t) = \left(\frac{d}{dt}\right)^{r} (I^{r-\alpha}_{w+}\theta)(t), \quad (r = [R(\alpha)] + 1;\ R(\alpha) > 0, \alpha \in \mathbb{C}), \tag{29}$$

where, $[t]$ is the greatest integer.

Hilfer [9] defined a generalized Riemann-Liouville fractional derivative operator $D^{\alpha,\beta}_{w+}$ of order $0 < \alpha < 1$ and type $0 \leq \beta \leq 1$ with respect to t as follows:

$$(D^{\alpha,\beta}_{w+}\theta)(t) = \left(I^{\beta(1-\alpha)}_{w+}\frac{d}{dt}\right)\left(I^{(1-\beta)(1-\alpha)}_{w+}\theta\right)(t), \qquad (r = [R(\alpha)]+1;\ R(\alpha) > 0, \alpha \in \mathbb{C}). \tag{30}$$

Theorem 8. *Suppose $w \in \mathbb{R}_+;\ \mu_i, \tau_i, \lambda_1, \lambda_2, \mu, \tau, \alpha, h \in \mathbb{C};\ R(\mu_i) > 0, R(\tau_i) > 0, R(\alpha) > 0,\ i = 1, ..., n$. Then, for $t > w$, we get the the following results:*

$$(i) \left(I^{\alpha}_{w+}\left[(a-w)^{\tau-1}\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(h(a-w)^{\mu};x)\right]\right)(t)$$
$$= (t-w)^{\tau+\alpha-1}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\ {}_2\Psi_{n+1}\left[\begin{matrix}(\tau,\mu),(1,1); \\ (\tau+\alpha,\mu),(\tau_i,\mu_i)_1^n;\end{matrix}\ h(t-w)^{\mu}\right]. \tag{31}$$

$$(ii) \left(D^{\alpha}_{w+}\left[(a-w)^{\tau-1}\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(h(a-w)^{\mu};x)\right]\right)(t)$$
$$= (t-w)^{\tau-\alpha-1}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\ {}_2\Psi_{n+1}\left[\begin{matrix}(\tau,\mu),(1,1); \\ (\tau-\alpha,\mu),(\tau_i,\mu_i)_1^n;\end{matrix}\ h(t-w)^{\mu}\right]. \tag{32}$$

and

$$(iii) \left(D^{\alpha,\beta}_{w+}\left[(a-w)^{\tau-1}\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(h(a-w)^{\mu};x)\right]\right)(t)$$
$$= (t-w)^{\tau-\alpha-1}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\ {}_2\Psi_{n+1}\left[\begin{matrix}(\tau,\mu),(1,1); \\ (\tau-\alpha,\mu),(\tau_i,\mu_i)_1^n;\end{matrix}\ h(t-w)^{\mu}\right]. \tag{33}$$

Proof. With the help of (10) and (28), the term by term fractional integration and the application of the result [24]

$$\left(I^{\mu}_{w+}\left[(a-w)^{\tau-1}\right]\right)(t) = \frac{\Gamma(\tau)}{\Gamma(\mu+\tau)}(a-w)^{\mu+\tau-1}, \quad (\mu,\tau \in \mathbb{C},\ R(\mu) > 0,\ R(\tau) > 0),$$

yields for $t > w$:

$$\left(I^{\alpha}_{w+}\left[(a-w)^{\tau-1}\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(h(a-w)^{\mu};x)\right]\right)(t)$$

$$=\left(I^{\alpha}_{w+}\left[\sum_{l=0}^{\infty}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\;\frac{h^l}{\prod\limits_{i=1}^{n}\Gamma(\mu_i l+\tau_i)}\,(a-w)^{\mu l+\tau-1}\right]\right)(t)$$

$$=\sum_{l=0}^{\infty}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\;\frac{h^l}{\prod\limits_{i=1}^{n}\Gamma(\mu_i l+\tau_i)}\left(I^{\alpha}_{w+}\left[(a-w)^{\mu l+\tau-1}\right]\right)(t)$$

$$=\sum_{l=0}^{\infty}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}$$

$$\times\;\frac{\Gamma(\mu l+\tau)\,h^l}{\Gamma(\mu l+\tau+\alpha)\prod\limits_{i=1}^{n}\Gamma(\mu_i l+\tau_i)}\;(t-w)^{\mu l+\tau+\alpha-1}$$

$$=(t-w)^{\tau+\alpha-1}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}$$

$$\times\sum_{l=0}^{\infty}\frac{\Gamma(\mu l+\tau)\,\Gamma(l+1)}{\Gamma(\mu l+\tau+\alpha)\,l!\prod\limits_{i=1}^{n}\Gamma(\mu_i l+\tau_i)}\;(h(t-w)^{\mu})^{l} \tag{34}$$

Now, using (7) and (8) in (34), we obtain the required result (31).

Next, by applying (10) and (29), we have,

$$\left(D^{\alpha}_{w+}\left[(a-w)^{\tau-1}\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(h(a-w)^{\mu};x)\right]\right)(t)$$

$$=\left(\frac{d}{dt}\right)^{r}\left(I^{r-\alpha}_{w+}\left[(a-w)^{\tau-1}\mathbf{E}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}_{(\mu_i,\tau_i)_1^n}(h(a-w)^{\mu};x)\right]\right)(t)$$

$$=\sum_{l=0}^{\infty}\frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l,\lambda_2-\lambda_1;x)}{B(\lambda_1,\lambda_2-\lambda_1)}\;\frac{\Gamma(\mu l+\tau)\,h^l}{\Gamma(\mu l+\tau+r-\alpha)\prod\limits_{i=1}^{n}\Gamma(\mu_i l+\tau_i)}$$

$$\times \left(\frac{d}{dt}\right)^r \left((t-w)^{\mu l+\tau+r-\alpha-1}\right),$$

employing (13) and (14), yields

$$= (t-w)^{\tau-\alpha-1} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l, \lambda_2-\lambda_1; x)}{B(\lambda_1, \lambda_2-\lambda_1)} \times \sum_{l=0}^{\infty} \frac{\Gamma(\mu l+\tau)\,\Gamma(l+1)}{\Gamma(\tau-\alpha+\mu l)\, l! \prod_{i=1}^{n} \Gamma(\mu_i l+\tau_i)} \, (h(t-w)^{\mu})^l. \tag{35}$$

Applying (7) and (8) in (35), we reach the result (32).

Lastly, from (10) and (30), we have

$$\left(D_{w+}^{\alpha,\beta}\left[(a-w)^{\tau-1}\mathbf{E}_{(\mu_i,\tau_i)_1^n}^{(\{j_m\}_{m\in\mathbb{N}_0});\lambda_1,\lambda_2}(h(a-w)^{\mu};x)\right]\right)(t)$$

$$= \left(D_{w+}^{\alpha,\beta}\left[\sum_{l=0}^{\infty} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l, \lambda_2-\lambda_1; x)}{B(\lambda_1, \lambda_2-\lambda_1)} \, \frac{h^l}{\prod_{i=1}^{n} \Gamma(\mu_i l+\tau_i)} \, (a-w)^{\mu l+\tau-1}\right]\right)(t)$$

$$= \sum_{l=0}^{\infty} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l, \lambda_2-\lambda_1; x)}{B(\lambda_1, \lambda_2-\lambda_1)} \, \frac{h^l}{\prod_{i=1}^{n} \Gamma(\mu_i l+\tau_i)} \left(D_{w+}^{\alpha,\beta}\left[(a-w)^{\mu l+\tau-1}\right]\right)(t). \tag{36}$$

Applying the result of Srivastava and Tomovski [24] in (36), yields

$$= (t-w)^{\tau-\alpha-1} \frac{B^{(\{j_m\}_{m\in\mathbb{N}_0})}(\lambda_1+l, \lambda_2-\lambda_1; x)}{B(\lambda_1, \lambda_2-\lambda_1)} \times \sum_{l=0}^{\infty} \frac{\Gamma(\mu l+\tau)\,\Gamma(l+1)}{\Gamma(\mu l+\tau-\alpha)\, l! \prod_{i=1}^{n} \Gamma(\mu_i l+\tau_i)} \, (h(t-w)^{\mu})^l \tag{37}$$

Finally using (7) and (8) in (37), we are getting the result (33).

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