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δ-DYNAMIC CHROMATIC NUMBER OF WHEEL GRAPH FAMILIES

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Abstract. An r-dynamic coloring of a graph G is a proper coloring c of the vertices such that $|c(N(v))| \geq \min\{r, d(v)\}$, for each $v \in V(G)$. The r-dynamic chromatic number of a graph G is the minimum k such that G has an r-dynamic coloring with k colors. In this paper, we obtain the r-dynamic chromatic number of middle, total, central and line graph of wheel graph.

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1. Introduction

Throughout this paper all graphs are finite and simple. The r-dynamic chromatic number was first introduced by Montgomery [11]. An r-dynamic coloring of a graph G is a map c from $V(G)$ to the set of colors such that (i) if $uv \in E(G)$, then $c(u) \neq c(v)$, and (ii) for each vertex $v \in V(G), |c(N(v))| \geq \min\{r, d(v)\}$, where $N(v)$ denotes the set of vertices adjacent to v and $d(v)$ its degree. The first condition characterizes proper colorings, the adjacency condition and second condition is double-adjacency condition. The r-dynamic chromatic number of a graph G, written $\chi_r(G)$, is the minimum k such that G has an r-dynamic proper k-coloring. The 1-dynamic chromatic number of a graph G is equal to its chromatic number. The 2-dynamic chromatic number of a graph has been studied under the name dynamic chromatic number in [1, 2, 3, 4, 7]. There are many upper bounds and lower bounds for $\chi_d(G)$ in terms of graph parameters. For example,

For a graph G with $\Delta(G) \geq 3$, Lai et al. [7] proved that $\chi_d(G) \leq \Delta(G) + 1$. An upper bound for the dynamic chromatic number of a d-regular graph G in terms of $\chi(G)$ and the independence number of G, $\alpha(G)$, was introduced in [5]. In fact, it was proved that $\chi_d(G) \leq \chi(G) + 2\log_2 \alpha(G) + 3$. Taherkhani gave in [12] an upper bound for $\chi_2(G)$ in terms of the chromatic number, the maximum degree Δ and the minimum degree δ. i.e., $\chi_2(G) - \chi(G) \leq \lceil (\Delta e)/\delta \log\left(2e\left(\Delta^2+1\right)\right)\rceil$.

Li et al. proved in [9] that the computational complexity of $\chi_d(G)$ for a 3-regular graph is an NP-complete problem. Furthermore, Liu and Zhou [8] showed that to determine whether there exists a 3-dynamic coloring, for a claw free graph with the maximum degree 3, is NP-complete.

In this paper, we study $\chi_r(G)$ when r is δ, the minimum degree of the graph. We find the δ-dynamic chromatic number for middle, total, central and line graph of wheel graph.

2. Main Results

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph [10] of G, denoted by $M(G)$ is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $M(G)$ are adjacent in $M(G)$ in case one of the following holds: (i) x, y are in $E(G)$ and x, y are adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$, and x, y are incident in G.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The total graph [10] of G, denoted by $T(G)$ is defined in the following way. The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y of $T(G)$ are adjacent in $T(G)$ in case one of the following holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y are adjacent in G. (iii) x is in $V(G), y$ is in $E(G)$, and x, y are incident in G.

The central graph [13] $C(G)$ of a graph G is obtained from G by adding an extra vertex on each edge of G, and then joining each pair of vertices of the original graph which were previously non-adjacent.

The line graph [6] of G denoted by $L(G)$ is the graph whose vertex set is the edge set of G. Two vertices of $L(G)$ are adjacent whenever the corresponding edges of G are adjacent.

For any integer $n \geq 4$, the wheel graph W_n is a graph of order n obtaining by joining a vertex to each of the $n-1$ vertices of a cycle C_{n-1}. Let $V(W_n) = \{v, v_1, v_2, \dots v_{n-1}\}$ and $E(W_n) = \{v_1v_2, v_2v_3, \dots, v_{n-2}v_{n-1}, v_{n-1}v_1\} \cup \{vv_1, \dots, vv_{n-1}\}$.

2.1. Middle graph of wheel

Theorem 1. *Let $n \geq 7$. The δ-dynamic chromatic number of the middle graph of a wheel of order n is $\chi_\delta(M(W_n)) = n$.*

Proof. Let $V(M(W_n)) = \{v, v_1, v_2, \dots, v_{n-1}\} \cup \{e_1, e_2, \dots, e_{n-1}\} \cup \{u_1, u_2, \dots, u_{n-1}\}$, where u_i is the vertex corresponding to the edge v_iv_{i+1} of W_n $(1 \leq i \leq n-2)$, $u_{n-1} =$

$v_{n-1}v_1$ and e_i is the vertex corresponding to the edge vv_i of $W_n\,(1 \leq i \leq n-1)$. By definition of the middle graph, the vertices v and $\{e_i : (1 \leq i \leq n-1)\}$ induce a clique of order K_n in $M(W_n)$. Thus, $\chi_\delta(M(W_n)) \geq n$.
Consider the following n-coloring of $M\,(W_n)$:
For $1 \leq i \leq n-1$, assign the color c_i to e_i and assign the color c_n to v. For $1 \leq i \leq n-1$, assign to vertex u_i one of the allowed colors - such color exists, because $deg(u_i) = 6$. For $1 \leq i \leq n-1$, if any, assign to vertex v_i one of the allowed colors - such color exists, because $deg(v_i) = 3$. Also, $\delta(G) = 3$. An easy check shows that $N(u)$ contains an induced clique of order 3, for every $u \in V\,(M\,(W_n))$. Thus, this coloring is a 3-dynamic coloring. Hence, $\chi_\delta(M(W_n)) \leq n$. Therefore, $\chi_\delta(M(W_n)) = n$, $\forall\ n \geq 7$.

2.2. Total graph of wheel

Theorem 2. *Let $n \geq 9$. The δ-dynamic chromatic number of the total graph of a wheel of order n is $\chi_\delta(T(W_n)) = n$.*

Proof. Let $V\,(T\,(W_n)) = \{v, v_1, v_2, \ldots, v_{n-1}\} \cup \{e_1, e_2, \ldots, e_{n-1}\} \cup \{u_1, u_2, \ldots, u_{n-1}\}$, where u_i is the vertex corresponding to the edge v_iv_{i+1} of $W_n\,(1 \leq i \leq n-2)$, $u_{n-1} = v_{n-1}v_1$ and e_i is the vertex corresponding to the edge vv_i of $W_n\,(1 \leq i \leq n-1)$. By the definition of the total graph, the vertices v and $\{e_i : (1 \leq i \leq n-1)\}$ induce a clique of order K_n in $T(W_n)$. Thus, $\chi_\delta(T(W_n)) \geq n$.
Consider the following n-coloring of $T\,(W_n)$:
For $1 \leq i \leq n-1$, assign color c_i to e_i and assign color c_n to v. We complete the coloring by assigning colors $\{c_1, c_2, \ldots, c_{n-1}\}$ to the remaining vertices consecutively as follows: color c_3 to v_1, c_4 to v_2, ... and color c_7 to u_1, c_8 to u_2, ... An easy check shows that this coloring is a 6-dynamic coloring. Hence, $\chi_\delta(T(W_n)) \leq n$. Therefore, $\chi_\delta(T(W_n)) = n$, $\forall\ n \geq 9$.

2.3. Central graph of wheel

Theorem 3. *Let $n \geq 4$. The δ-dynamic chromatic number of the central graph of a wheel of order n is $\chi_\delta(C(W_n)) = n$.*

Proof. Let $V\,(C\,(W_n)) = \{v, v_1, v_2, \ldots, v_{n-1}\} \cup \{e_1, e_2, \ldots, e_{n-1}\} \cup \{u_1, u_2, \ldots, u_{n-1}\}$, where u_i is the vertex corresponding to the edge v_iv_{i+1} of $W_n\,(1 \leq i \leq n-2)$, $u_{n-1} = v_{n-1}v_1$ and e_i is the vertex corresponding to the edge vv_i of $W_n\,(1 \leq i \leq n-1)$. Clearly, the graph induced by $\{v_{2i} :\ i = 1, 2, \ldots, \lfloor (n-1)/2 \rfloor\}$ is a complete graph. Thus, a proper coloring assign at least $\lfloor (n-1)/2 \rfloor$ colors to them. The same happens with the subgraph induced by $\{v_{2i-1} :\ i = 1, 2, \ldots, \lfloor (n-1)/2 \rfloor\}$. Moreover, if we are considering a δ-dynamic coloring when $n-1$ is odd v_{n-1} should have a different

color from v_{2i-1}, $i = 1, 2, \dots, (n-1)/2$, because v_{n-1} and v_1 are the only neighbors of u_{n-1}, and v_{n-1} is adjacent to v_{2i-1}, $i = 2, \dots, (n-1)/2$. A similar reasoning also shows that in a δ-dynamic coloring, the colors assigned to odd vertices should be different to the colors assigned to even vertices and that all of them should be different from the color assigned to v. Thus, $\chi_\delta(C(W_n)) \geq n$.
Consider the following n-coloring of $C(W_n)$:
For $1 \leq i \leq n-1$, assign the color c_i to v_i and assign the color c_n to v. For $1 \leq i \leq n-1$, assign to vertex u_i and e_i one of the allowed colors - such color exists, because $\deg(u_i) = \deg(e_i) = 2$. An easy check shows that this coloring is a δ-dynamic coloring. Hence, $\chi_\delta(C(W_n)) \leq n$. Therefore, $\chi_\delta(C(W_n)) = n$.

2.4. Line graph of wheel

Theorem 4. *Let $n \geq 6$. The δ-dynamic chromatic number of the line graph of a wheel of order n is $\chi_\delta(L(W_n)) = n - 1$.*

Proof. Let $V(L(W_n)) = \{e_1, e_2, \dots, e_{n-1}\} \cup \{u_1, u_2, \dots, u_{n-1}\}$, where u_i is the vertex corresponding to the edge $v_i v_{i+1}$ of W_n $(1 \leq i \leq n-2)$, $u_{n-1} = v_{n-1}v_1$ and e_i is the vertex corresponding to the edge vv_i of W_n $(1 \leq i \leq n-1)$. By definition of the line graph, $\{e_1, e_2, \dots, e_{n-1}\}$ induces a clique of order K_{n-1} in $L(W_n)$. Thus, $\chi_\delta(L(W_n)) \geq n - 1$.
Consider the following $n-1$-coloring of $L(W_n)$:
For $1 \leq i \leq n-1$, assign the color c_i to e_i. We complete the coloring by assigning colors $\{c_1, c_2, \dots, c_{n-1}\}$ to the remaining vertices consecutively as follows: color c_4 to u_1, c_5 to u_2, ... An easy check shows that this coloring is a 4-dynamic coloring. Hence, $\chi_\delta(L(W_n)) \leq n-1$. Therefore, $\chi_\delta(L(W_n)) = n - 1$.

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