Acta Universitatis Apulensis — No. 77/2024
ISSN: 1582-5329 — pp. 25-38
http://www.uab.ro/auajournal/ — doi: 10.17114/j.aua.2024.77.03

CHARACTERISING SMOOTHNESS OF TYPE A SCHUBERT VARIETIES THROUGH PALINDROMIC POINCARÉ POLYNOMIAL METHOD

A.P. Adetunji, H.P. Adeyemo, D.O.A. Ajayi, S.A. Ilori

Abstract. The aim of this paper is to give a characterisation for smoothness of type A Schubert varieties in terms of the exponents of their monomials. We extend Smoothness of Schubert varieties in type A, from S_n to $\mathbb{Z}_n^+$. As a consequence, we give examples to support our results.

2010 *Mathematics Subject Classification*: 2020 Mathematics Subject Classification: Primary: 14M15, 14B05. Secondary: 32M05

Keywords: schubert varieties, smoothness, singularity, poincare polynomials, palindromic.

1. Introduction

Schubert varieties are singular algebraic subvarieties of the flag varieties, indexed by permutation matrices and studied in different types by means of linear algebra. The Schubert varieties are the closure of the Schubert cells which form additive basis that generate the cohomology ring. The basis for the cohomology ring are the geometric and the algebraic basis. The Schubert polynomials are the geometric basis while the monomials are the algebraic basis.

In [7], Lakshmibai and Seshadri, determined the singularities of Schubert varieties by computing the set of points for which the Schubert varieties are singular. Smoothness and singularities of Schubert varieties were also determined in [6] using permutation pattern avoidance for the elements of the symmetric group. Infact they describe this as the 4231 and 3412 permutation pattern avoidance.

In a paper of [1], Carrel determined smoothness of type A Schubert varieties using the symmetric group of n letters. We extend the result of [1] in type A from S_n to $\mathbb{Z}_n^+$. This is done in terms of the exponents of the monomials of the Schubert varieties.

In Section 2 we give the basic definitions needed to proof our theorem. In Section 3 we considerer the main results which is the characterisation of smoothness using the exponents of the monomials of the Schubert varieties using the method of Palindromic Poincare polynomials. Therefore the following problems arises naturally: *Can we extend the smoothness of Schubert varieties in type A to a more general group of n letters*? It is the purpose of this paper to give answers to this question in the affirmative.

We shall first discuss the Schubert varieties as the subvarieties of the flag varieties.

Let $V = \mathbb{C}^n$, which denotes a complex vector space of dimension n, A flag $V_\bullet$ in $\mathbb{C}^n$ is a sequence of ordered subspaces,

$$V_\bullet : V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V \tag{1}$$

$\ni dim_{\mathbb{C}} V_i = i$ where $0 \le i \le n$.

The set of all such flags forms a smooth complex projective variety called the full flag variety and denoted by $\mathcal{F}\ell_n(\mathbb{C})$ [2].

Remark 1. *(i) The set of all k dimensional subspaces of an n dimensional vector space, denoted by $Gr(k,n)$, is called the Grassmannian variety which is an example of a flag variety.*

(ii) Flag varieties are smooth complex projective varieties because they can be embedded into the products of the Grassmannians which are embedded into the products of higher dimensional projective spaces by means of the Plücker embedding map.

$$\mathcal{F}\ell_n(\mathbb{C}) \hookrightarrow \prod_{k=1}^{n-1} Gr(k,n) \hookrightarrow \prod_{k=1}^{n-1} \mathbb{P}^{\begin{pmatrix} n \\ k \end{pmatrix} - 1}. \tag{2}$$

(iii) Flag varieties are seen to be homogeneous spaces since it has a transitive group action and it is identified with the quotient group G/B also since for any $V_\bullet \in \mathcal{F}\ell_n(\mathbb{C})$ and $g \in Gl_n(\mathbb{C}) \ni gV_\bullet = V'_\bullet \in \mathcal{F}\ell_n(\mathbb{C})$.

(iv) Flag varieties are compact homogeneous spaces because, there is an action of the closed compact subgroup of $Gl_n(\mathbb{C})$ known as the Unitary group $U_n(\mathbb{C})$ on it. The action results in $\mathcal{F}\ell_n(\mathbb{C})$ becoming a compact homogeneous space with dimension $\frac{n(n-1)}{2}, \forall n \in \mathbb{N}$.

The $T - fixed$ points are Flags associated to permutation matrices. Given that σ is a permutation in S_n, then the T-fixed point of a flag $V_\bullet$ is

$$V_\bullet^\sigma = \langle e_{\sigma(1)} \rangle \subset \langle e_{\sigma(1)} e_{\sigma(2)} \rangle \subset \cdots \subset \langle e_{\sigma(1)} e_{\sigma(2)} \cdots e_{\sigma(n)} \rangle \tag{3}$$

defined by

$$V_\bullet^\sigma \mapsto \sigma B = \{\sigma B : \sigma \in G\}, \tag{4}$$

where $B = \{a_{ij} \in Gl(n, \mathbb{C}) \ni a_{ij} = 0, \forall i > j\}$ is the Borel subgroup of the general linear group G and σ is a permutation matrix. There are $n!$ of these permutation matrices.

The elements of $\mathcal{F}\ell_n(\mathbb{C})^T$ embeds in $\mathcal{F}\ell_n(\mathbb{C})$ as the set of the T-fixed points. The normalizer of T on G and $N_G(T)/T$ consist of the monomial matrices with only one non-zero entry in each row and each column $\mathcal{F}\ell_n(\mathbb{C})^T \cong W \cong S_n$ where W is the Whyl group [2].

Example 1. *Let $\sigma = 2413$ where $\sigma \in S_n$. The T-fixed point of the flag $V_\bullet^\sigma$ is*

$$V_\bullet^\sigma = \langle e_{\sigma(2)} \rangle \subset \langle e_{\sigma(2)} e_{\sigma(4)} \rangle \subset \langle e_{\sigma(2)} e_{\sigma(4)} e_{\sigma(1)} \rangle \subset \langle e_{\sigma(2)} e_{\sigma(4)} e_{\sigma(1)} e_{\sigma(3)} \rangle. \tag{5}$$

The elements of the symmetric groups index B-orbits $n!$ flag variety G/B and they form the well known Bruhat decomposition theorem.

Theorem 1 ([3])**.** *The general linear group $G = Gl_n(\mathbb{C})$ is a disjoint union $G = \coprod_{\sigma \in W} B\sigma B$.*

The flag varieties are partitioned into cells arising from double Cosets, that is

$$\mathcal{F}\ell_n(\mathbb{C}) = G/B = \coprod_{\sigma \in S_n} B\sigma B/B = \coprod_{\sigma \in S_n} C_\sigma \tag{6}$$

called the Bruhat cell. Each Bruhat cell $C_\sigma \cong \mathbb{C}^{l(\sigma)}$ where $\mathbb{C}^{l(\sigma)}$ is the affine space and $l(\sigma)$ is the length of σ. The length of σ is given by the number of inversions or the no of transpositions or reflections of the permutation. The transitive closure of the Schubert cells is known as the Schubert varieties denoted by

$$X_\sigma = \bar{C_\sigma} = \bigcup_{v \leq \sigma} C_v, \tag{7}$$

where $v \leq \sigma$ defines a partial order on $W \cong S_n$ called the Bruhat order [2]. Hence $l(v) \leq l(\sigma)$.

Example 2. *For the X_σ where σ is the permutation of S_4, we show that we have a Schubert variety .*

Applying the definition of Schubert varieties we have

$$X_{4321} = \bigcup_{v,(4321),\in\mathbb{Z}_n^+,v\leq(4321)} C_v, \tag{8}$$

$$\begin{aligned} X_{4321} = C_{4321} \bigcup C_{4312} \bigcup C_{4231} \bigcup C_{3421} \bigcup C_{4132} \bigcup C_{4213} \bigcup C_{3214} \bigcup C_{2431} \\ \bigcup C_{3241} \bigcup C_{1432} \bigcup C_{4123} \bigcup C_{2413} \bigcup C_{3142} \bigcup C_{3214} \bigcup C_{2314} \bigcup C_{1423} \\ \bigcup C_{1342} \bigcup C_{2143} \bigcup C_{3124} \bigcup C_{2134} \bigcup C_{1243} \bigcup C_{1324} \bigcup C_{2314} \bigcup C_{1234}. \end{aligned}$$

Remark 2. *(i) The Schubert varieties X_σ and its dual X^σ are irreducible subvarieties of the flag varieties $\mathcal{F}\ell_n(\mathbb{C})$ of dimension $l(\sigma)$ and $n-l(\sigma)$ respectively.*

(ii) The dimension of the flag variety is related to the dimension of X_σ and X^σ by $dim(X_\sigma + X^\sigma) = dim\mathcal{F}\ell_n(\mathbb{C})$.

The type A Schubert variety is said to be smooth if it is rationally smooth that is if for all $v, \sigma \in X_\sigma$ the Poincare polynomials of the variety is equal one

The classes of the closure of the Schubert cells forms additive basis for the cohomology of $\mathcal{F}\ell_n(\mathbb{C})$. The homology of the flag varieties does not have a ring structure but since the flag varieties $\mathcal{F}\ell_n(\mathbb{C})$ satisfies Poincare duality, this implies that there exist an isomorphism from the homology to the cohomology of $\mathcal{F}\ell_n(\mathbb{C})$ given by the map ,

$$f : H_{(n-k)}(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z}) \to H^k(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z}) \tag{9}$$

and defined by

$$f[X_\sigma] = [X^\sigma] \in H^k(\mathcal{F}\ell_n(\mathbb{C})) \tag{10}$$

called the Schubert class [4].

The Poincaré map f enables one to identify each graded piece of the cohomology ring $H^k(\mathcal{F}\ell_n(\mathbb{C});$
$\mathbb{Z})$ with the homology group $H_{n-k}(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z})$. Thus, the Schubert classes forms additive $\mathbb{Z}$ basis that generates the cohomology ring $H^k(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z})$. The basis for the cohomology ring are the geometric basis and the algebraic basis. The degree of $[X_\sigma]$ is $2\dim[X_\sigma] = 2l(\sigma)$.

The $k^{th}-$ Betti number, $b_k = \dim^{2k}(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z})$, $0 \leq k \leq dim\mathcal{F}\ell_n(\mathbb{C})$. That is the number of generators of each of the graded piece of the cohomology ring $\mathcal{F}\ell_n(\mathbb{C})$ gives b_k. The algebraic basis for the cohomology of the ring $\mathcal{F}\ell_n(\mathbb{C})$ is described as follows:

Definition 1 ([4]). *A Symmetric function of a polynomial ring $\mathbb{Z}[x_1, x_2, \cdots, x_n]$ in $x_1, x_2, \cdots, x_n$ variables over an integral domain $\mathbb{Z}$ is symmetric if it is invariant for every permutation $e_i \in S_n$.*

Proposition 1 ([4]). *The cohomology ring $H^{2l(\sigma)}(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z})$ is generated by the basic classes $x_1, \cdots, x_n$ subject to the relations $e_i(x_1, \cdots, x_n) = 0$ for $1 \leq i \leq n$. The classes $x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m}$ with exponents $i_j \leq m-j$ form a $\mathbb{Z}$ basis for $H^{2l(\sigma)}(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z})$.*

The flag varieties are generated by the basic classes with generators x_1, x_2, x_3, x_4 as in the following:

Example 3. *The $H^{2l(\sigma)}(\mathcal{F}\ell_n(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, \cdots, x_n]/I$, for $I = \langle e_i(x_1, \cdots, x_n) \rangle$, where $1 \leq i \leq n$ and e_i is the ith elementary symmetric function. For $\mathcal{F}\ell_n(\mathbb{C}) = V_6$, $H^{2l(\sigma)}(\mathcal{F}\ell_4(\mathbb{C}); \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, x_3, x_4]/I = \langle e_1, e_2, e_3, e_4 \rangle$ since the cohomology ring is a graded ring it implies that,*

$$H^{2k}(\mathcal{F}\ell_4(\mathbb{C}); \mathbb{Z}) = \bigoplus_{k=0}^{n} H^{2k}(\mathcal{F}\ell_4(\mathbb{C}); \mathbb{Z}). \tag{11}$$

where $0 \leq k \leq 6$.

Observe that:

- *For $k = 0, H^{2k}(\mathcal{F}\ell_4(\mathbb{C}); \mathbb{Z}) = H^{2.0} = 1$.*
- *For $k = 1, H^{2k}(\mathcal{F}\ell_4(\mathbb{C}); \mathbb{Z}) = H^{2.1} = \langle x_1, x_2, x_3 \rangle$.*
- *For $k = 2, H^{2k}(\mathcal{F}\ell_4(\mathbb{C}); \mathbb{Z}) = H^{2.2} = \langle x_1^2, x_2^2, x_1x_3, x_1x_2, x_2x_3 \rangle$.*
- *For $k = 3, H^{2k}(\mathcal{F}\ell_4(\mathbb{C}); \mathbb{Z}) = H^{2.3} = \langle x_1^3, x_1^2x_2, x_2^2x_1, x_1x_2x_3, x_1^2x_3, x_2^2x_3 \rangle$.*
- *For $k = 4, H^{2k}(\mathcal{F}\ell_4(\mathbb{C}); \mathbb{Z}) = H^{2.4} = \langle x_1^3x_2, x_1^3x_3, x_1^2x_2^2, x_1^2x_2x_3, x_1x_2^2x_3 \rangle$.*
- *For $k = 5, H^{2k}(\mathcal{F}\ell_4(\mathbb{C}); \mathbb{Z}) = H^{2.5} = \langle x_1^3x_2^2, x_1^3x_2x_3, x_1^2x_2^2x_3 \rangle$.*
- *For $k = 6, H^{2k}(\mathcal{F}\ell_4(\mathbb{C}); \mathbb{Z}) = H^{2.6} = \langle x_1^3x_2^2x_3 \rangle$.*

.

2. The Palindromic Poincaré Polynomial Method and Kazhdan-Lusztig Polynomials

We start this section with some definitions and theorems of interest. The Poincaré polynomials which was first used in ([1]) to determine smoothness and singularities of Schubert varieties.

Definition 2 ([5])**.** *For a complex algebraic variety X, its Poincaré polynomial is given by*

$$P_x(t) = \sum_{i \geq 0} dim_{\mathbb{C}}(H^i(X))t^i \tag{12}$$

where $H^i(X)$ is the singular homology of X.

Definition 3 ([2])**.** *The Poincaré polynomial of a Schubert variety (X_σ) is said to be the rank generating function for the interval $[id, \sigma]$, where the rank is the number of inversions then $P_\sigma(t) = \sum_{v \leq \sigma} t^{l(v)}$ where the sum is over all elements $v \leq \sigma$ in the Bruhat-Chevalley order on W where W is the wehl group.*

If W containing a set $S \in (W, S)$ is a coxeter system then W is a coxeter group. The Wehl group is a coxeter group. It was mentioned in [1] that the Poincare polynomial is Palindromic (also see [2])

A Poincaré polynomial $p(t) = v_0 + v_1 t + \cdots + v_r t^r$ is Palindromic if for $v \leq \sigma$ and $l(v) \leq l(\sigma)$ then $p(t) = t^r p(t^{-1})$.

Theorem 2. *[5] Let (W, S) be an arbitrary Coxeter system. For $v \leq y \leq \sigma$*

$$\sharp\{r \in R | v \leq ry \leq \sigma\} \geq l(\sigma) - l(v). \tag{13}$$

A result on the notion of a smooth Palindromic polynomial is as stated in [1]

Theorem 3. *[1] For any permutation $\sigma \in S_n$ the Schubert variety X_σ is smooth if and only if the Poincaré polynomial is Palindromic.*

The Kazhdan-Lusztig polynomial is a polynomial in one variable that has the following properties:

1. $P_{v,\sigma}(t) = 0$ if $v \leq \sigma$;
2. The number of edges connected to $P_{v,\sigma}(t)$ is less or equal to $\frac{1}{2}(l(\sigma) - l(v) - 1)$;
3. $P_{\sigma,\sigma}(t) = 1$.
4. $P_{v,\sigma}(t) \neq 0 \leftrightarrow v \leq \sigma$.

The following are equivalent for any $v \leq \sigma$ in W [6]

1. X_σ is rationally smooth at e_v,
2. $P_{x,\sigma}(t) = 1$ for all $v \leq x \leq \sigma$.

Theorem 4. *[8] Let $IH(\sigma)$ be the intersection cohomology sheaf of X_σ with respect to middle perversity, then*

1. $P_{v,\sigma}(t) = \sum dim(IH^{2i}(X_\sigma)v)q^i$ *which implies that the coefficients of* $P_{v,\sigma}(t)$ *are nonnegative;*

2. $P_{v,\sigma}(t)t^{l(v)} = \sum_{v\leq\sigma} dim(IH^{2i}(X_\sigma))q^i$ *Which implies palindromic symmetric;*

3. $P_{v,\sigma}(t) = 1$ *for every* $v \leq \sigma$ *if and only if* X_σ *is rationally smooth. and this will be taken to be the definition for rational smoothness.*

3. Main Results

Theorem 5. *Let* $\sigma \in \mathbb{Z}_+^n$ *be the monomial exponent of the* X_σ*, then the following are equivalent:*

(i) The Schubert variety X_σ *is rationally smooth at every point.(since smoothness in type A is equivalent to rational smoothness);*

(ii) The Poincaré polynomial $P_\sigma(t)$ *is Palindromic;*

(iii) The Bruhat graph $\Gamma(id, \sigma)$ *is regular, that is every vertex has the same number of edges,* $l(\sigma)$*.*

Proof. We show

For the case $i \Rightarrow ii$

Suppose X_σ is rationally smooth at every point then we must show that the Poincaré polynomial is symmetric.

As $X(\sigma)$ is rationally smooth,

$$P_{v,\sigma}(t) = 1, \forall, v \leq \sigma. \tag{14}$$

From the definition (2)

$$P_\sigma(t) = \sum_i dimH^{2i}(X(\sigma))t^i = \sum_{v\leq\sigma} t^{l(v)}P_{v,\sigma}(t) \tag{15}$$

which is a Palindromic polynomial.

Since $P_{v,\sigma}(t) = 1, \forall, v \leq \sigma$ we have,

$$P_\sigma(t) = \sum_{v\leq\sigma} t^{l(v)}P_{v,\sigma}(t) = \sum_{v\leq\sigma} t^{l(v)} \tag{16}$$

is Palindromic.

Next we show that $(ii) \Rightarrow (iii)$

Assume $P_\sigma(t)$ is symmetric then we must show that every vertex has the same number of edges $l(\sigma)$.

Since $P_\sigma(t)$, is Palindromic, then

$$t^{l(\sigma)}P_\sigma(t^{-1}) = P_\sigma(t). \tag{17}$$

$$t^{l(\sigma)}\sum_{v\leq\sigma} t^{-l(v)} = P_\sigma(t) = \sum_{v\leq\sigma} t^{l(v)}. \tag{18}$$

i.e

$$\sum_{v\leq\sigma}(t^{l(\sigma)-l(v)} - t^{l(v)}) = 0. \tag{19}$$

Taking the derivative of (19), we have

$$\sum_{v\leq\sigma}[(l(\sigma)-l(v))t^{l(\sigma)-l(v)-1} - l(v)t^{l(v)-1}] = 0. \tag{20}$$

For $t = 1$, (20) becomes

$$\sum_{v\leq\sigma}(l(\sigma)-l(v)) = \sum_{v\leq\sigma} l(v). \tag{21}$$

Now, let $v \in W$, by Theorem 2, $l(v) = \sharp\{r \in R, |rv < v\}$
i.e.

$$\sum_{v\leq\sigma} l(v) = \sum_{v\leq\sigma}\sharp\{r\in R, |rv < v\} = \sum_{v\leq\sigma}\sharp\{r \in R, |v < rv \leq \sigma\} \tag{22}$$

From Deodhar's Inequality, [5] we have that

$$\sharp\{r \in R, |x \leq ry \leq \sigma\} \geq l(\sigma) - l(x), \forall, x \leq y \leq \sigma. \tag{23}$$

If $x = y$,

$$\sharp\{r \in R, |y \leq ry \leq \sigma\} \geq l(\sigma) - l(y), \forall y \leq \sigma. \tag{24}$$

Thus (22) becomes

$$\sum_{v\leq\sigma} l(v) = \sum_{v\leq\sigma}\sharp\{r \in R, |v < rv \leq \sigma\} \geq \sum_{v\leq\sigma} l(\sigma) - l(v) = \sum_{v\leq\sigma} l(v). \tag{25}$$

$$\sum_{v\leq\sigma} l(\sigma) - l(v) = \sum_{v\leq\sigma}\sharp\{r \in R, |v < rv \leq \sigma\}. \tag{26}$$

Hence,

$$l(\sigma) = l(v) + \sharp\{r \in R, |v < rv \leq \sigma\}, \forall, v \leq \sigma. \tag{27}$$

$$= \text{number of edges of vertex } v \leq \sigma.$$

Next, we show that

Suppose that every vertex of $\Gamma(id, \sigma)$ has the same number $l(\sigma)$ of edges, then we must show that X_σ is rationally smooth at every point. That is $P_{v,\sigma}(t) = 1$ For $v \leq \sigma$

We show by induction on $l(\sigma) - l(v) = k$ and by the definition of smoothness. For $k = 0, 1, 2, 3$

From (16)

$$\frac{d}{dt}[t^{l(\sigma)-l(v)} P_{v,\sigma}(t^{-2})]_{t=1} = \sum_{r \in R | v < rv \leq \sigma} P_{rv,\sigma}(1) \tag{28}$$

i.e.

$$\frac{d}{dt}[t^3(1 + \frac{\alpha}{t^2})]_{t=1} = \sum_{r \in R | v < rv \leq \sigma} P_{rv,\sigma}(1) = 3 + \alpha. \tag{29}$$

Now, for $r \in R$ and $v < rv \leq \sigma$. Observe that

$$\begin{gathered} l(v) < l(rv) \\ l(v) \leq l(rv) - 1 \\ -l(rv) \leq -l(v) - 1 \\ l(\sigma) - l(rv) \leq l(\sigma) - l(v) - 1 = 3 - 1 = 2. \end{gathered}$$

By definition of smoothness of type A Schubert varieties, $P_{rv,\sigma}(t) = 1$, for $r \in R$ such that $v < rv \leq \sigma$ and so

$$\sum_{r \in R | v < rv \leq \sigma} P_{rv,\sigma}(1) = \sum_{r \in R | v < rv \leq \sigma} (1) = \sharp\{r \in R | v < rv \leq \sigma\} \tag{30}$$

$$= l(\sigma) - l(v) = 3. \tag{31}$$

Combining (29) and (31) we have

$$3 + \alpha = 3, \tag{32}$$

i.e

$$\alpha = 0. \tag{33}$$

Hence

$$P_{v,\sigma}(t) = 1 + \alpha t = 1, \forall t. \tag{34}$$

Assume that $P_{v,\sigma} = 1$ is true for all $l(\sigma) - l(v) \leq k - 1$. For some $k \geq 1$, we want to show that $P_{v,\sigma} = 1$.

For $l(\sigma) - l(v) = k$.

Let

$$f(t) = t^{l(\sigma)-l(v)}[P_{v,\sigma}(t^{-2}) - 1] \tag{35}$$

Where $l(\sigma) - l(v) = k \geq 1$, so $v < \sigma$

It is easy to see that, $P_{v,\sigma}(t)$ has degree $\frac{1}{2}(l(\sigma) - l(v) - 1)$, and $P_{v,\sigma}(0) = 1$

i.e,

$$P_{v,\sigma}(t) = \sum_{i=0}^{\frac{1}{2}(l(\sigma)-l(v)-1)} \alpha_i t^i \tag{36}$$

with $\alpha_0 = P_{v,\sigma}(0) = 1$.

So

$$f(t) = t^{l(\sigma)-l(v)}[\sum_{i=0}^{\frac{1}{2}(l(\sigma)-l(v)-1)} \alpha_i t^{-2i} - 1] \tag{37}$$

i.e.

$$t^{l(\sigma)-l(v)} \sum_{i=1}^{\frac{1}{2}(l(\sigma)-l(v)-1)} \alpha_i t^{-2i} = \sum_{i=1}^{\frac{1}{2}(k-1)} \alpha_i t^{k-2i}, \tag{38}$$

where $k = l(\sigma) - l(v)$.

Observe that

$1 \leq i \leq \frac{1}{2}(k-1) \Rightarrow 2 \leq 2i \leq (k-1) \Rightarrow 1-k \leq -2i \leq -2 \Rightarrow 1 \leq k-2i \leq (k)$

Hence, $f(t)$ is a polynomial with no constant term.

By Deodhar inequality [5], and differentiating with respect to t at $t = 1$ we have

$$\frac{d}{dt}[t^{l(\sigma)-l(v)} P_{v,\sigma}(t^{-2})]_{t=1} = \sum_{r \in R | v < rv \leq \sigma} P_{rv,\sigma}(1). \tag{39}$$

i.e

$$f'(1) = \sum_{r \in R | v < rv \leq \sigma} P_{rv,\sigma}(1) - [l(\sigma) - l(v)]. \tag{40}$$

Let $r \in R$ be such that $v < rv \leq \sigma$,

$$\begin{aligned} v < rv &\Rightarrow l(v) < l(rv) \\ &\Rightarrow l(v) \leq l(rv) - 1 \\ &\Rightarrow -l(rv) \leq -l(v) - 1 \\ &\Rightarrow l(\sigma) - l(rv) \leq l(\sigma) - l(v) - 1 = k - 1. \end{aligned}$$

So from the induction hypothesis $P_{rv,\sigma}(1) = 1$, we have

$$f'(1) = \sum_{r \in R | v < rv \leq \sigma} 1 - [l(\sigma) - l(v)]. \tag{41}$$

i.e.

$$\sharp\{r \in R | v < rv \leq \sigma\} - [l(\sigma) - l(v)] = 0. \tag{42}$$

From (37) and (38) we have

$$f(t) = \sum_{i=1}^{\frac{1}{2}(k-1)} \alpha_i t^{k-2i} \tag{43}$$

i.e

$$P_{v,\sigma}(t) = \sum_{i=0}^{\frac{1}{2}(k-1)} \alpha_i t^i \tag{44}$$

and so

$$f^{'}(1) = \sum_{i=1}^{\frac{1}{2}(k-1)} \alpha_i (k-2i) = 0. \tag{45}$$

The coefficients α_i of the Kazhdan-Lusztig polynomials [5] are non negative and $k - 2i \geq 1, \forall i$

Hence $\alpha_i = 0, \forall i$ So, $f(t) = 0, \forall t$ i.e.

$$t^{l(\sigma)-l(v)}[P_{v,\sigma}(t^{-2}) - 1] = 0, \forall t \tag{46}$$

i.e.

$$P_{v,\sigma}(t) = 1, \forall t. \tag{47}$$

which shows that the Schubert variety X_σ is rationally smooth at every vertex

Hence smoothness in type A.

Example 4. *For the Schubert variety where σ is the exponent of the monomials of the X_σ for the permutation of S_4, we have the Bruhat order.*

Observe that

$$P_\sigma(\mathcal{F}\ell_4(\mathbb{C}), t) = t^6 + 3t^5 + 5t^4 + 6t^3 + 5t^2 + 3t + 1. \tag{48}$$

$$(1 \quad 3 \quad 5 \quad 6 \quad 5 \quad 3 \quad 1).$$

Clearly, $\mathcal{F}\ell_4(\mathbb{C})$ is smooth.

Example 5. *For the X_σ where σ is the exponent of the monomials of the X_σ for the permutation of S_4, we Show that X_σ is smooth if it is palindromic .*

Applying the definition of Schubert varieties we have

$$X_{3,2,1} = \bigcup_{v,(3,2,1),\in\mathbb{Z}_n^+, v\leq(3,2,1)} C_v \tag{49}$$

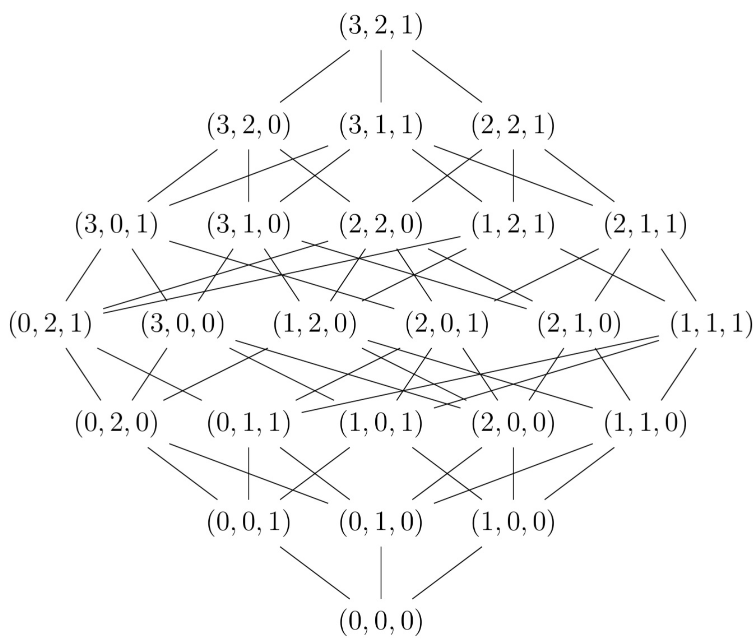

Figure 1: Bruhat graph for the exponents of the monomials of X_σ

$$X_{3,2,1} = C_{3,2,1} \bigcup C_{3,2,0} \bigcup C_{3,1,1} \bigcup C_{2,2,1} \bigcup C_{3,0,1} \bigcup C_{3,1,0} \bigcup C_{2,2,0} \bigcup C_{1,2,1}$$
$$\bigcup C_{2,1,1} \bigcup C_{0,2,1} \bigcup C_{3,0,0} \bigcup C_{1,2,0} \bigcup C_{2,0,1} \bigcup C_{2,1,0} \bigcup C_{1,1,1} \bigcup C_{0,2,0}$$
$$\bigcup C_{0,1,1} \bigcup C_{1,0,1} \bigcup C_{2,0,0} \bigcup C_{1,1,0} \bigcup C_{0,0,1} \bigcup C_{0,1,0} \bigcup C_{1,1,0} \bigcup C_{0,0,0}.$$

The Poincaré polynomial of the Schubert variety $X_{3,2,1}$ is given by

$$P_{3,2,1}(t) = 1 + 3t + 5t^2 + 6t^3 + 5t^4 + 3t^5 + t^6. \tag{50}$$

Hence, the Schubert variety is smooth since the Poincaré polynomial is Palindromic.

Remark 3. *When showing smoothness and singularity of Schubert variety using the exponent of its monomials ;*

(i) Smoothness is understood in terms of the exponents of the monomial of the Schubert variety.

(ii) The sum of each exponent of a monomial term gives the length of the Schubert variety.

(iii) *The addition of the exponent term on same row gives the coefficient of the Poincaré polynomial.*

(iv) *The sum of the exponent terms are reducing as we move down the bruhat order.*

We have successfully shown smoothness of type A Schubert varieties using the exponents of the monomials of theSchubert varieties. Thus extends the underlying group from S_n to $\mathbb{Z}_n^+$ in paper of [1] and references therein.

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A.P. ADETUNJI,
Department of Mathematics Faculty of Science,
University of Lagos,

Akoka, Yaba,Lagos, Nigeria
email*padetunji@unilag.edu.ng*

H.P. Adeyemo Department of Mathematics
University of Ibadan
Ibadan, Oyo, Nigeria
email*ph.adeyemo@ui.edu.ng*

D.O.A. Ajayi Department of Mathematics
University of Ibadan
Ibadan, Oyo, Nigeria
email*adelaideajayi@yahoo.com*

S. A. Ilori Department of Mathematics
University of Ibadan
Ibadan, Oyo, Nigeria
email*ilorisa1@yahoo.com*