

APPLICATION OF DISCRETE PROBABILITY DISTRIBUTION SERIES FOR ANALYTIC FUNCTIONS

A.A. THOMBRE, D.N. CHATE

ABSTRACT. The main objective of the present paper is to investigate the discrete probability distribution series for the function classes of analytic functions in the open unit disk. Furthermore, we consider an integral operator related to the discrete probability distribution series.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: Analytic univalent functions, discrete probability distribution, probability mass function, convolution operator.

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic and univalent in the open unit disk given by

$$\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}. \quad (2)$$

Let $\mathcal{T}$ be the subclass of $\mathcal{A}$ consisting of functions whose non zero coefficient of the form second on, given by

$$f(z) = z - \sum_{n=2}^{\infty} |A_n| z^n. \quad (3)$$

In 2018, [5] introduce the discrete probability distribution whose probability mass function is

$$\zeta(n) = \frac{a_n}{T}, \quad n = 0, 1, 2, 3, \dots \quad (4)$$

where $T = \sum_{n=0}^{\infty} a_n$ is convergent series for $a_n \geq 0, \forall n \in N$. Also, we introduce the series

$$\varphi(p) = \sum_{n=0}^{\infty} a_n p^n. \quad (5)$$

It is easy to see that the series given by (5) is convergent for $|p| < 1$ and $p = 1$. Further, [5] introduce a power series whose coefficients are probabilities of the generalized distribution for functions as follows:

$$K_{\varphi}(z) = z + \sum_{n=2}^{\infty} \frac{a_{n-1}}{T} z^n. \quad (6)$$

Further, we define the following function:

$$PK_{\varphi}(z) = z - \sum_{n=2}^{\infty} \frac{a_{n-1}}{T} z^n. \quad (7)$$

The Hadamard product (or convolution) of two power series $f(z) = \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=2}^{\infty} b_n z^n$ is defined as

$$(f * g)(z) = (g * f)(z) = \sum_{n=2}^{\infty} a_n b_n z^n.$$

Next, we introduce the convolution operator $PK_{\varphi}(f, z)$ for function f of the form (3) as follows

$$PK_{\varphi}(f, z) = PK_{\varphi}(z) * f(z) = z - \sum_{n=2}^{\infty} |A_n| \frac{a_{n-1}}{T} z^n. \quad (8)$$

Let $\mathcal{S}^*(\alpha, \beta)$ be the subclass of $\mathcal{T}$ consisting of functions which satisfy the condition

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1 - 2\alpha} \right| < \beta, \quad z \in \mathbb{U}, \quad (9)$$

where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$.

Also, let $\mathcal{C}^*(\alpha, \beta)$ be the subclass of $\mathcal{T}$ consisting of functions which satisfy the condition

$$\left| \frac{\frac{zf''(z)}{f'(z)} - 1}{\frac{zf''(z)}{f'(z)} + 2(1 - \alpha)} \right| < \beta, \quad z \in \mathbb{U}, \quad (10)$$

where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$.

The classes $\mathcal{S}^*(\alpha, \beta)$ and $\mathcal{C}^*(\alpha, \beta)$, were introduced by [7]. Also, we note that for $\beta = 1$ these classes reduces $\mathcal{S}^*(\alpha, 1) = \mathcal{S}^*(\alpha)$ and $\mathcal{C}^*(\alpha, 1) = \mathcal{K}(\alpha)$, the class of starlike and convex functions of order α ($0 \leq \alpha < 1$) [3, 4].

A function $f \in \mathcal{A}$ is said to in the class $\mathcal{R}^\tau(C, D)$, if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(C - D)\nu - D[f'(z) - 1]} \right| < 1, \quad (z \in \mathbb{U}) \quad (11)$$

where $\nu \in \mathbb{C} \setminus \{0\}$, $-1 \leq C < D \leq 1$. This class was introduced by [1].

Lemma 1. [7] *A function $f(z)$ of the form (3) is in $\mathcal{S}^*(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} [n(1 + \beta) - 1 + \beta(1 - 2\alpha)] |a_n| \leq 2\beta(1 - \alpha). \quad (12)$$

Lemma 2. [7] *A function $f(z)$ of the form (3) is in $\mathcal{C}^*(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} n[n(1 + \beta) - 1 + \beta(1 - 2\alpha)] |a_n| \leq 2\beta(1 - \alpha). \quad (13)$$

Lemma 3. [1] *If $f \in \mathcal{R}^\tau(C, D)$ is of the form (1) then*

$$|a_n| = (C - D) \frac{|\nu|}{n}, \quad n \in \mathbb{N} \setminus \{1\}. \quad (14)$$

Motivated with works of [2, 5, 6, 8, 9, 10, 11, 12], we obtain necessary and sufficient conditions for $PK_\varphi(z)$ and $PK_\varphi(f, z)$ in the classes $\mathcal{S}^*(\alpha, \beta)$ and $\mathcal{C}^*(\alpha, \beta)$. Also, we obtain inclusion relations for aforecited classes with $\mathcal{R}^\tau(C, D)$.

2. Necessary and Sufficient Conditions

Theorem 4. *If $PK_\varphi(z)$ that is of the form (7) is in the class $\mathcal{S}^*(\alpha, \beta)$, if and only if*

$$\frac{1}{T} \left[(1 + \beta)\varphi'(1) + 2\beta(1 - \alpha)[\varphi(1) - \varphi(0)] \right] \leq 2\beta(1 - \alpha). \quad (15)$$

Proof.

$$PK_\varphi(z) = z - \sum_{n=2}^{\infty} \frac{a_{n-1}}{T} z^n,$$

in view of Lemma 1, it is sufficient to show that

$$\sum_{n=2}^{\infty} [n(1+\beta) - 1 + \beta(1-2\alpha)] \left| \frac{a_{n-1}}{T} \right| \leq 2\beta(1-\alpha).$$

Now

$$\begin{aligned} \varrho_1(\alpha, \beta, T) &= \sum_{n=2}^{\infty} [n(1+\beta) - 1 + \beta(1-2\alpha)] \left| \frac{a_{n-1}}{T} \right| \\ &= \frac{1}{T} \left[(1+\beta) \sum_{n=2}^{\infty} (n-1)a_{n-1} + 2\beta(1-\alpha) \sum_{n=2}^{\infty} a_{n-1} \right] \\ &= \frac{1}{T} \left[(1+\beta) \sum_{n=1}^{\infty} na_n + 2\beta(1-\alpha) \sum_{n=1}^{\infty} a_n \right] \\ &= \frac{1}{T} \left[(1+\beta)\varphi'(1) + 2\beta(1-\alpha)[\varphi(1) - \varphi(0)] \right] \\ &\leq 2\beta(1-\alpha). \end{aligned} \tag{16}$$

This completes the proof of Theorem.

Theorem 5. *If $PK_{\varphi}(z)$ that is of the form (7) is in the class $\mathcal{C}^*(\alpha, \beta)$, if and only if*

$$\frac{1}{T} \left[(1+\beta)\varphi''(1) + 2[1+3(2-\alpha)]\varphi'(1) + 2\beta(1-\alpha)[\varphi(1) - \varphi(0)] \right] \leq 2\beta(1-\alpha). \tag{17}$$

Proof. Since

$$PK_{\varphi}(z) = z - \sum_{n=2}^{\infty} \frac{a_{n-1}}{T} z^n,$$

in view of Lemma 2, it is sufficient to show that

$$\sum_{n=2}^{\infty} n[n(1+\beta) - 1 + \beta(1-2\alpha)] \left| \frac{a_{n-1}}{T} \right| \leq 2\beta(1-\alpha). \tag{18}$$

Now

$$\begin{aligned}
\varrho_2(\alpha, \beta, T) &= \sum_{n=2}^{\infty} n[n(1+\beta) - 1 + \beta(1-2\alpha)] \left| \frac{a_{n-1}}{T} \right| \\
&= \frac{1}{T} \left[(1+\beta) \sum_{n=2}^{\infty} (n-1)(n-2)a_{n-1} \right. \\
&\quad \left. + 2(1+3(2-\alpha)) \sum_{n=2}^{\infty} (n-1)a_{n-1} + 2\beta(1-\alpha) \sum_{n=2}^{\infty} a_{n-1} \right] \\
&= \frac{1}{T} \left[(1+\beta)\varphi''(1) + 2(1+3(2-\alpha))\varphi'(1) + 2\beta(1-\alpha)[\varphi(1) - \varphi(0)] \right] \\
&\leq 2\beta(1-\alpha). \tag{19}
\end{aligned}$$

This completes the proof of Theorem.

3. Inclusion Properties

Theorem 6. *If $f \in \mathcal{R}^\tau(C, D)$ is of the form (3) and the operator $PK_\varphi(f, z)$, defined by (8) in the class $\mathcal{C}^*(\alpha, \beta)$, if and only if*

$$\frac{(C-D)|\nu|}{T} [(n-1)\varphi'(1) + 2\beta(1-\alpha)(\varphi(1) - \varphi(0))] \leq 2\beta(1-\alpha). \tag{20}$$

Proof. In view of Lemma 2, it is sufficient to show that

$$\varrho_3(\alpha, \beta, T) = \sum_{n=2}^{\infty} n[n(1+\beta) - 1 + \beta(1-2\alpha)] \left| \frac{a_{n-1}}{T} \right| \leq 2\beta(1-\alpha).$$

Since $f \in \mathcal{R}^\tau(C, D)$, then by Lemma 3 we have

$$|a_n| = \frac{(C-D)|\nu|}{n}.$$

Therefore

$$\begin{aligned}
\varrho_3(\alpha, \beta, T) &\leq \frac{(C-D)|\nu|}{T} \sum_{n=2}^{\infty} [n(1+\beta) - 1 + \beta(1-2\alpha)] a_{n-1} \\
&= \frac{(C-D)|\nu|}{T} \sum_{n=2}^{\infty} [(n-1)(1+\beta) + 2\beta(1-\alpha)] a_{n-1} \\
&= \frac{(C-D)|\nu|}{T} [(n-1)\varphi'(1) + 2\beta(1-\alpha)(\varphi(1) - \varphi(0))] \\
&\leq 2\beta(1-\alpha). \tag{21}
\end{aligned}$$

This completes the proof of Theorem.

4. An Integral Operator

In this section we introduce an integral operator $PI_\varphi(z)$ as follows:

$$PI_\varphi(z) = \int_0^z \frac{PK_{\varphi(t)}}{t} dt, \quad (22)$$

and we obtain a necessary and sufficient condition for $PI_\varphi(z)$ belonging to the class $\mathcal{C}^*(\alpha, \beta)$.

Theorem 7. *If $PK_\varphi(z)$ is defined by (7), then $PI_\varphi(z)$ defined by (22) in the class $\mathcal{C}^*(\alpha, \beta)$, if and only if (15) satisfied.*

Proof. Since

$$PI_\varphi(z) = z - \sum_{n=2}^{\infty} \frac{a_{n-1}}{nT} z^n,$$

in view of Lemma 2, it is sufficient to show that

$$\sum_{n=2}^{\infty} n[n(1+\beta) - 1 + \beta(1-2\alpha)] \left| \frac{a_{n-1}}{nT} \right| \leq 2\beta(1-\alpha).$$

Now

$$\begin{aligned} \varrho_4(\alpha, \beta, T) &= \sum_{n=2}^{\infty} n[n(1+\beta) - 1 + \beta(1-2\alpha)] \left| \frac{a_{n-1}}{nT} \right| \\ &= \frac{1}{T} \left[(1+\beta) \sum_{n=2}^{\infty} (n-1)a_{n-1} + 2\beta(1-\alpha) \sum_{n=2}^{\infty} a_{n-1} \right] \\ &= \frac{1}{T} \left[(1+\beta)\varphi'(1) + 2\beta(1-\alpha)[\varphi(1) - \varphi(0)] \right] \\ &\leq 2\beta(1-\alpha). \end{aligned} \quad (23)$$

This completes the proof of Theorem.

REFERENCES

- [1] K. Dixit and S. Pal, *On a class of univalent functions related to complex order*, Indian J. Pure Appl. Math., 26(9) (1995), 889-896.
- [2] S. S. Eker, G. Murugusundaramoorthy and B. Şeker, *Spiral-like functions associated with Miller–Ross-type Poisson distribution series*, Bol. Soc. Mat. Mex. 29, 16 (2023).

- [3] V. Gupta and P. Jain, *Certain classes of univalent functions with negative coefficients*, Bull. Aust. Math. Soc., 14, 409–416.
- [4] P. N. Kamble, M. G. Shrigan and H. M. Srivastava, *A novel subclass of univalent functions involving operators of fractional calculus*, Int. J. Appl. Math., 30(6) (2017), 501–514.
- [5] S. Porwal, *Generalized distribution and its geometric properties associated with univalents functions*, Journal of Complex Analysis, Art. ID 8654506:5 pages.
- [6] S. Porwal, A. L. Pathak and O. Mishra, *Wright distribution and its applications on univalent functions*, U.P.B. Sci. Bull., Series A 84 (4) (2022), 81–88.
- [7] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Soc., 51 (1975), 109–116.
- [8] M. G. Shrigan, *Subclasses of univalent functions involving Touchard polynomials*, Acta Univ. Apulensis, 71 (2020), 65–76
- [9] M. G. Shrigan and S. Yalçın, *Spiral-like functions associated with Wright distribution series*, Analele Univ. Oradea Fasc. Matematica, Tom XXXI (1) (2024), 19–29.
- [10] M. G. Shrigan, S. Yalçın and S. Altinkaya, *Unified approach to starlike and convex functions involving Poisson distribution series*, Bul. Acad. Ştiinţe Repub. Moldova Mat., 97 (3) (2021), 11–20.
- [11] M. G. Shrigan, G. Murugusundaramoorthy and T. Bulboacă, *Classes of analytic functions associated with the (p, q) -derivative operator for generalized distribution satisfying subordinate condition*, TWMS J. Appl. Eng. Math., 14 (3), 2024, 1311–1327.
- [12] M. G. Shrigan, S. D. Bhourgubde and G. D. Shelake, *Some applications of a Wright distribution series on subclasses of univalent functions*, Stud. Univ. Babeş-Bolyai Math. (to appear)

Acknowledgements. The authors would like to thank the referees for their careful reading and helpful comments.

A. A. Thombre
Research Scholar, Department of Mathematics,
Swami Ramanand Teerth Marathawada University,
Nanded, India
email: ashokthombre123@gmail.com

D. N. Chate
Department of Mathematics,
Sanjeevane Mahavidyalaya,
Chapoli, India
email: dhananjayachate@gmail.com