Acta Universitatis Apulensis No. 73/2023
ISSN: 1582-5329 pp. 17-31
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2023.73.02

ON THE BICOMPLEX PADOVAN AND BICOMPLEX PERRIN NUMBERS

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Abstract. In this paper we first introduce the bicomplex Padovan and bicomplex Perrin numbers which generalize Padovan and Perrin numbers, and then we derive the Binet-like formulas, the generating functions and the exponential generating functions, series, sums of these sequences. Also, we obtain some binomial identities for them.

2010 *Mathematics Subject Classification*: 11B39, 05A15, 11R52, 97F50.

Keywords: Padovan numbers, Perrin numbers, bicomplex Padovan sequence, bicomplex Perrin sequence, Binet-like formula, Generating function, Binomial sum.

1. Introduction

Quaternion numbers were defined by Hamilton in 1843. These numbers have an algebraic structure that has all the properties of real and complex numbers, except the property of change of multiplication. There are also many studies on quaternion numbers [1, 2, 7, 9, 11, 12]. The bicomplex numbers were defined by James Cockle [6]. The quaternions were defined by Hamilton in 1943 as an extension to the complex numbers. Cockle defined a bicomplex number as $a = m_1 + jm_2$ using the new unit j, which Hamilton described, inspired by the definition of quaternions. Segre contributed different interpretations by working again on the algebra of bicomplex numbers in 1892 [29]. Srivastava has created various studies using complex numbers [5, 22]. Recently, some studies have been done on bicomplex numbers. Relevant sources can be consulted for some of the studies carried out [3, 4, 13, 14, 18, 23, 24, 25, 26, 33, 34, 35, 36, 37]. In addition, there are some studies on bicomplex numbers and their algebra, geometry, topology, dynamic and quantum properties. For some of these studies can look at [10, 15, 16, 19, 20, 21, 27, 28, 30]. They are formed a four dimensional real vector space with a multiplicative operation. They have played a significant role in physical science, differential geometry, analysis and synthesis of mechanisms and machines, theory of relativity and others. Unlike quaternion algebra, the bicomplex contains the commutative form.

A bicomplex is defined by

$$q = q_0 + q_1 i + q_2 j + q_3 ij$$

where q_0, q_1, q_2 and q_3 are real numbers. The bicomplex multiplication is defined using the rules;

$$i^2 = -1, \quad j^2 = -1, \quad \text{and} \quad ij = ji.$$

Bicomplex bir sayı aşağıdaki biçimde ifade edilebilir:

$$q = q_0 + q_1 i + q_2 j + q_3 ij = q_0 + q_1 i + (q_2 + q_3 i)j$$

Special number sequences have play important role in mathematics and applied sciences. Moreover, some special number sequences such as Fibonacci, Lucas, Pell, Jacobsthal, Padovan and Perrin sequences have many applications in art, music, photography, architecture, painting, engineering, geometry and others. It is well-known that the term golden ratio is defined the ratio of two consecutive Fibonacci numbers converges to

$$\frac{1+\sqrt{5}}{2} \approx 1.618034.$$

The golden ratio has many applications in engineering, physics, architecture, arts and other. In similar way, the ratio of two consecutive Padovan or Perrin numbers converges to

$$\sqrt[3]{\frac{1}{2} + \frac{1}{6}\sqrt{\frac{23}{3}}} + \sqrt[3]{\frac{1}{2} - \frac{1}{6}\sqrt{\frac{23}{3}}} \approx 1.324718,$$

that is called as "plastic ratio". The plastic ratio (number) was first defined by Gerard Cordonnier in 1924. He described applications to architecture and illustrated the use of the plastic number in many buildings. Furthermore, the plastic number is the unique real root of the equation

$$x^3 - x - 1 = 0,$$

the characteristic equation of Padovan number sequences. (see [11, 17, 32]). The Padovan sequence $\{P_n\}_{n\geq 0}$ is defined by the initial values $P_0 = P_1 = P_2 = 1$ and the recurrence relation

$$P_{n+3} = P_{n+1} + P_n, \qquad \text{for all } n \geq 0. \tag{1}$$

First few terms of this sequence are $1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28$. The Perrin sequence $\{R_n\}_{n\geq 0}$ is defined by the initial values $R_0 = 3, R_1 = 0$ and $R_2 = 2$ and the recurrence relation

$$R_{n+3} = R_{n+1} + R_n, \qquad \text{for all } n \geq 0. \tag{2}$$

First few terms of Perrin sequence are $3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29$. Padovan and Perrin sequence can be found in [17, 31, 32]. For every $x \in \mathbb{N}$, one can write the Binet-like formulas for the Padovan and Perrin sequences as the form

$$P_n = a\alpha^n + b\beta^n + c\gamma^n \tag{3}$$

and

$$R_n = \alpha^n + \beta^n + \gamma^n \tag{4}$$

where α, β and γ are the roots of the characteristic equation

$$x^3 - x - 1 = 0 \tag{5}$$

associated with (1) and (2), where

$$a = \frac{(\beta-1)(\gamma-1)}{(\alpha-\beta)(\alpha-\gamma)}, b = \frac{(\alpha-1)(\gamma-1)}{(\beta-\alpha)(\beta-\gamma)}, c = \frac{(\alpha-1)(\beta-1)}{(\alpha-\gamma)(\beta-\gamma)}.$$

The Binet-like formulas for the Padovan and Perrin sequences were given in [7, 8, 9].

2. The Bicomplex Padovan and Perrin Sequences

In this section, we define two new bicomplex sequences which are the bicomplex Padovan and bicomplex Perrin sequences. Then, we give their Binet-like formulas, generating functions, series functions, partial sums and certain binomial sums.

Definition 1. *The bicomplex Padovan sequence $\{\mathcal{C}P_n\}_{n\geq 0}$ is defined by*

$$\mathcal{C}P_n = P_n + P_{n+1}i + P_{n+2}j + P_{n+3}ij \tag{6}$$

where P_n is the nth Padovan number.

Definition 2. *The bicomplex Perrin sequence $\{\mathcal{C}R_n\}_{n\geq 0}$ is defined by*

$$\mathcal{C}R_n = R_n + R_{n+1}i + R_{n+2}j + R_{n+3}ij \tag{7}$$

where R_n is the nth Perrin number.

Theorem 1 (Binet-like formula)**.** *The Binet-like formulas for the bicomplex Padovan sequence* $\{\mathcal{C}P_n\}_{n\geq 0}$ *is*

$$\mathcal{C}P_n = a\hat{\alpha}\alpha^n + b\hat{\beta}\beta^n + c\hat{\gamma}\gamma^n, \tag{8}$$

where

$$\hat{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 ij,$$
$$\hat{\beta} = 1 + \beta i + \beta^2 j + \beta^3 ij,$$

and

$$\hat{\gamma} = 1 + \gamma i + \gamma^2 j + \gamma^3 ij.$$

Proof. From the definition of nth bicomplex Padovan sequence $\{\mathcal{C}P_n\}$ in (6) and Binet-like formula for the nth Padovan number P_n, we write

$$\begin{aligned}
\mathcal{C}P_n &= P_n + P_{n+1}i + P_{n+2}j + P_{n+3}ij \\
&= (a\alpha^n + b\beta^n + c\gamma^n) + (a\alpha^{n+1} + b\beta^{n+1} + c\gamma^{n+1})i + (a\alpha^{n+2} \\
&\quad + b\beta^{n+2} + c\gamma^{n+2})j + (a\alpha^{n+3} + b\beta^{n+3} + c\gamma^{n+3})ij \\
&= a(1 + \alpha i + \alpha^2 j + \alpha^3 ij)\alpha^n + b(1 + \beta i + \beta^2 j + \beta^3 ij)\beta^n \\
&\quad + c(1 + \gamma i + \gamma^2 j + \gamma^3 ij)\gamma^n \\
&= a\hat{\alpha}\alpha^n + b\hat{\beta}\beta^n + c\hat{\gamma}\gamma^n
\end{aligned}$$

Thus, the proof is completed.

Theorem 2 (Binet-like formula)**.** *The Binet-like formula for the bicomplex Perrin sequence* $\{\mathcal{C}R_n\}_{n\geq 0}$ *is*

$$\mathcal{C}R_n = \hat{\alpha}\alpha^n + \hat{\beta}\beta^n + \hat{\gamma}\gamma^n, \tag{9}$$

where

$$\hat{\alpha} = 1 + \alpha i + \alpha^2 j + \alpha^3 ij,$$
$$\hat{\beta} = 1 + \beta i + \beta^2 j + \beta^3 ij,$$

and

$$\hat{\gamma} = 1 + \gamma i + \gamma^2 j + \gamma^3 ij.$$

Proof. From the definition of nth bicomplex Perrin sequence $\{\mathcal{C}R_n\}$ in (7) and Binet-like formula for the nth Perrin number R_n, we write

$$\begin{aligned}
\mathcal{C}R_n &= R_n + R_{n+1}i + R_{n+2}j + R_{n+3}ij \\
&= (\alpha^n + \beta^n + \gamma^n) + (\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1})i + (\alpha^{n+2} + \beta^{n+2} \\
&\quad + \gamma^{n+2})j + (\alpha^{n+3} + \beta^{n+3} + \gamma^{n+3})ij \\
&= (1 + \alpha i + \alpha^2 j + \alpha^3 ij)\alpha^n + (1 + \beta i + \beta^2 j + \beta^3 ij)\beta^n \\
&\quad + (1 + \gamma i + \gamma^2 j + \gamma^3 ij)\gamma^n \\
&= \hat{\alpha}\alpha^n + \hat{\beta}\beta^n + \hat{\gamma}\gamma^n
\end{aligned}$$

Thus, the proof is completed.

Theorem 3. *The generating function for the bicomplex Padovan sequence $\{\mathcal{C}P_n\}$ is*

$$\mathcal{G}_{\mathcal{C}P}(x) = \frac{1 + i + j + 2ij + (1 + i + 2j + 2ij)x + (i + j + ij)x^2}{1 - x^2 - x^3}.$$

Proof. Assume that the function

$$\mathcal{G}_{\mathcal{C}P}(x) = \sum_{n=0}^{\infty} \mathcal{C}P_n x^n = \mathcal{C}P_0 + \mathcal{C}P_1 x + \mathcal{C}P_2 x^2 + \mathcal{C}P_3 x^3 + \ldots + \mathcal{C}P_n x^n + \ldots$$

be generating function of the bicomplex Padovan sequence. Multiply both of side of the equality by the term $-x^2$ such as

$$-x^2 \mathcal{G}_{\mathcal{C}P}(x) = -\mathcal{C}P_0 x^2 - \mathcal{C}P_1 x^3 - \mathcal{C}P_2 x^4 - \mathcal{C}P_3 x^5 - \ldots - \mathcal{C}P_n x^{n+2} - \ldots$$

and multiply by the term $-x^3$ such as

$$-x^3 \mathcal{G}_{\mathcal{C}P}(x) = -\mathcal{C}P_0 x^3 - \mathcal{C}P_1 x^4 - \mathcal{C}P_2 x^5 - \mathcal{C}P_3 x^6 - \ldots - \mathcal{C}P_n x^{n+3} - \ldots$$

Then, we write

$$\begin{aligned}
(1 - x^2 - x^3)\mathcal{G}_{\mathcal{C}P}(x) =& \mathcal{C}P_0 + \mathcal{C}P_1 x + (\mathcal{C}P_2 - \mathcal{C}P_0)x^2 + (\mathcal{C}P_3 - \mathcal{C}P_1 \\
&- \mathcal{C}P_0)x^3 + \ldots + (\mathcal{C}P_n - \mathcal{C}P_{n-2} - \mathcal{C}P_{n-3})x^n + \ldots
\end{aligned}$$

Now, by using

$$\mathcal{C}P_0 = 1 + i + j + 2ij,$$

$$\mathcal{C}P_1 = 1 + i + 2j + 2ij,$$

$$\mathcal{C}P_2 = 1 + 2i + 2j + 3ij,$$

and

$$\mathcal{C}P_n - \mathcal{C}P_{n-2} - \mathcal{C}P_{n-3} = 0,$$

we obtain that

$$\mathcal{G}_{\mathcal{C}P}(x) = \frac{1 + i + j + 2ij + (1 + i + 2j + 2ij)x + (i + j + ij)x^2}{1 - x^2 - x^3}$$

Thus, the proof is completed.

Theorem 4. *The generating function of the bicomplex Perrin sequence* $\{\mathcal{C}R_n\}$ *is*

$$\mathcal{G}_{\mathcal{C}R}(x) = \frac{3 + 2j + 3ij + (2i + 3j + 2ij)x + (-1 + 3i + 2ij)x^2}{1 - x^2 - x^3}$$

Proof. Let

$$\mathcal{G}_{\mathcal{C}R}(x) = \sum_{n=0}^{\infty} \mathcal{C}R_n x^n = \mathcal{C}R_0 + \mathcal{C}R_1 x + \mathcal{C}R_2 x^2 + \mathcal{C}R_3 x^3 + \ldots + \mathcal{C}R_n x^n + \ldots$$

be generating function of the bicomplex Perrin sequence. Now multiply both of side of the equality by term $-x^2$ such as

$$-x^2 \mathcal{G}_{\mathcal{C}R}(x) = -\mathcal{C}R_0 x^2 - \mathcal{C}R_1 x^3 - \mathcal{C}R_2 x^4 - \mathcal{C}R_3 x^5 - \ldots - s\mathcal{C}R_n x^{n+2} - \ldots$$

and multiply by $-x^3$ such as

$$-x^3 \mathcal{G}_{\mathcal{C}R}(x) = -\mathcal{C}R_0 x^3 - \mathcal{C}R_1 x^4 - \mathcal{C}R_2 x^5 - \mathcal{C}R_3 x^6 - \ldots - \mathcal{C}R_n x^{n+3} - \ldots.$$

Then, we write

$$\begin{aligned}(1 - x^2 - x^3)\mathcal{G}_{\mathcal{C}R}(x) =& \mathcal{C}R_0 + \mathcal{C}R_1 x + (\mathcal{C}R_2 - \mathcal{C}R_0)x^2 + (\mathcal{C}R_3 - \mathcal{C}R_1 \\ & - \mathcal{C}R_0)x^3 + \ldots + (\mathcal{C}R_n - \mathcal{C}R_{n-2} - \mathcal{C}R_{n-3})x^n + \ldots\end{aligned}$$

By using

$$\mathcal{C}R_0 = 3 + 2j + 3ij,$$
$$\mathcal{C}R_1 = 2i + 3j + 2ij,$$
$$\mathcal{C}R_2 = 2 + 3i + 2j + 5ij,$$

and

$$\mathcal{C}R_n - \mathcal{C}R_{n-2} - \mathcal{C}R_{n-3} = 0,$$

we obtain that

$$\mathcal{G}_{\mathcal{C}R}(x) = \frac{3 + 2j + 3ij + (2i + 3j + 2ij)x + (-1 + 3i + 2ij)x^2}{1 - x^2 - x^3}.$$

This completes the proof.

Theorem 5. *The exponential generating function for the bicomplex Padovan sequence $\{\mathcal{C}P_n\}$ is*

$$E_{\mathcal{C}P}(x) = ae^{\alpha x} + be^{\beta x} + ce^{\gamma} = \sum_{n=0}^{\infty} \frac{\mathcal{C}P_n}{n!} x^n.$$

Proof. We know that,

$$e^{\alpha x} = \sum_{n=0}^{\infty} \frac{\alpha^n x^n}{n!}, \quad e^{\beta x} = \sum_{n=0}^{\infty} \frac{\beta^n x^n}{n!}, \quad e^{\gamma x} = \sum_{n=0}^{\infty} \frac{\gamma^n x^n}{n!}$$

Multiplying each side of the identities, respectively, by a, b and c and adding of them, we obtain that

$$ae^{\alpha x} + be^{\beta x} + ce^{\gamma x} = \sum_{n=0}^{\infty} (a\alpha^n + b\beta^n + c\gamma^n) \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{\mathcal{C}P_n}{n!} x^n.$$

Theorem 6. *The exponential generating function for the bicomplex Perrin sequence $\{\mathcal{C}R_n\}$ is*

$$E_{\mathcal{C}R}(x) = e^{\alpha x} + e^{\beta x} + e^{\gamma} = \sum_{n=0}^{\infty} \frac{\mathcal{C}R_n}{n!} x^n.$$

Proof. We know that,

$$e^{\alpha x} = \sum_{n=0}^{\infty} \frac{\alpha^n x^n}{n!}, \quad e^{\beta x} = \sum_{n=0}^{\infty} \frac{\beta^n x^n}{n!}, \quad e^{\gamma x} = \sum_{n=0}^{\infty} \frac{\gamma^n x^n}{n!}$$

we obtain that

$$e^{\alpha x} + e^{\beta x} + e^{\gamma x} = \sum_{n=0}^{\infty} (\alpha^n + \beta^n + \gamma^n) \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{\mathcal{C}R_n}{n!} x^n.$$

Theorem 7. *The series function for the bicomplex Padovan sequence $\{\mathcal{C}P_n\}$ is*

$$\mathcal{S}_{\mathcal{C}P}(x) = \frac{(1+i+j+2ij)x^3 + (1+i+2j+2ij)x^2 + (i+j+ij)x}{x^3 - x - 1}.$$

Proof. Assume that the function

$$\mathcal{S}_{\mathcal{C}P}(x) = \sum_{n=0}^{\infty} \mathcal{C}P_n x^{-n} = \mathcal{C}P_0 + \mathcal{C}P_1 x^{-1} + \mathcal{C}P_2 x^{-2} + \mathcal{C}P_3 x^{-3} + \ldots + \mathcal{C}P_n x^{-n} + \ldots$$

be series function of the bicomplex Padovan sequence. Multiply both of side of the equality by the term x^3 such as

$$x^3 \mathcal{S}_{\mathcal{C}P}(x) = \mathcal{C}P_0 x^3 + \mathcal{C}P_1 x^2 + \mathcal{C}P_2 x + \mathcal{C}P_3 + \ldots + \mathcal{C}P_n x^{3-n} + \ldots$$

and multiply by the term $-x$ such as

$$-x\mathcal{C}_{\mathcal{C}P}(x) = -\mathcal{C}P_0 x - \mathcal{C}P_1 - \mathcal{C}P_2 x^{-1} - \mathcal{C}P_3 x^{-2} - \ldots - \mathcal{C}P_n x^{1-n} - \ldots$$

Then, we write

$$\begin{aligned}(x^3 - x - 1)\mathcal{S}_{\mathcal{C}P}(x) =& \mathcal{C}P_0 x^3 + \mathcal{C}P_1 x^2 + (\mathcal{C}P_2 - \mathcal{C}P_0)x + (\mathcal{C}P_3 - \mathcal{C}P_1 - \mathcal{C}P_0) \\ &+ \ldots + (\mathcal{C}P_n - \mathcal{C}P_{n-2} - \mathcal{C}P_{n-3})x^{3-n} + \ldots\end{aligned}$$

Now, by using

$$\mathcal{C}P_0 = 1 + i + j + 2ij,$$
$$\mathcal{C}P_1 = 1 + i + 2j + 2ij,$$
$$\mathcal{C}P_2 = 1 + 2i + 2j + 3ij,$$

and

$$\mathcal{C}P_n - \mathcal{C}P_{n-2} - \mathcal{C}P_{n-3} = 0,$$

we obtain that

$$\mathcal{S}_{\mathcal{C}P}(x) = \frac{(1 + i + j + 2ij)x^3 + (1 + i + 2j + 2ij)x^2 + (i + j + ij)x}{x^3 - x - 1}$$

Thus, the proof is completed.

Theorem 8. *The series function for the bicomplex Perrin sequence* $\{\mathcal{C}R_n\}$ *is*

$$\mathcal{S}_{\mathcal{C}R}(x) = \frac{(3 + 2j + 3ij)x^3 + (2i + 3j + 2ij)x^2 + (-1 + 3i + 2ij)x}{x^3 - x - 1}.$$

Proof. Assume that the function

$$\mathcal{S}_{\mathcal{C}R}(x) = \sum_{n=0}^{\infty} \mathcal{C}R_n x^{-n} = \mathcal{C}R_0 + \mathcal{C}R_1 x^{-1} + \mathcal{C}R_2 x^{-2} + \mathcal{C}R_3 x^{-3} + \ldots + \mathcal{C}R_n x^{-n} + \ldots$$

be series function of the bicomplex Perrin sequence. Multiply both of side of the equality by the term x^3 such as

$$x^3\mathcal{S}_{\mathcal{C}R}(x) = \mathcal{C}R_0x^3 + \mathcal{C}R_1x^2 + \mathcal{C}R_2x + \mathcal{C}R_3 + \ldots + \mathcal{C}R_nx^{3-n} + \ldots$$

and multiply by the term $-x$ such as

$$-x\mathcal{C}_{\mathcal{C}R}(x) = -\mathcal{C}R_0x - \mathcal{C}R_1 - \mathcal{C}R_2x^{-1} - \mathcal{C}R_3x^{-2} - \ldots - \mathcal{C}R_nx^{1-n} - \ldots$$

Then, we write

$$\begin{aligned}(x^3 - x - 1)\mathcal{S}_{\mathcal{C}R}(x) =&\mathcal{C}R_0x^3 + \mathcal{C}R_1x^2 + (\mathcal{C}R_2 - \mathcal{C}R_0)x + (\mathcal{C}R_3 - \mathcal{C}R_1 - \mathcal{C}R_0)\\ &+ \ldots + (\mathcal{C}R_n - \mathcal{C}R_{n-2} - \mathcal{C}R_{n-3})x^{3-n} + \ldots\end{aligned}$$

By using

$$\begin{aligned}\mathcal{C}R_0 &= 3 + 2j + 3ij,\\ \mathcal{C}R_1 &= 2i + 3j + 2ij,\\ \mathcal{C}R_2 &= 2 + 3i + 2j + 5ij,\end{aligned}$$

and

$$\mathcal{C}R_n - \mathcal{C}R_{n-2} - \mathcal{C}R_{n-3} = 0,$$

we obtain that

$$\mathcal{S}_{\mathcal{C}R}(x) = \frac{(3 + 2j + 3ij)x^3 + (2i + 3j + 2ij)x^2 + (-1 + 3i + 2ij)x}{x^3 - x - 1}$$

Thus, the proof is completed.

Theorem 9. *The partial sum of the first n terms of the bicomplex Padovan sequence* $\{\mathcal{C}P_n\}$ *is*

$$\sum_{i=0}^{n} \mathcal{C}P_i = \mathcal{C}P_{n+5} - 2 - 3i - 4j - 5ij, \quad n \geq 0.$$

Proof. We know that

$$\mathcal{C}P_{n+3} = \mathcal{C}P_{n+1} + \mathcal{C}P_n$$

So, applying to the identity above, we deduce that

$$\begin{aligned}\mathcal{C}P_3 &= \mathcal{C}P_1 + \mathcal{C}P_0,\\ \mathcal{C}P_4 &= \mathcal{C}P_2 + \mathcal{C}P_1,\\ \mathcal{C}P_5 &= \mathcal{C}P_3 + \mathcal{C}P_2,\\ &\ldots,\\ \mathcal{C}P_{n+1} &= \mathcal{C}P_{n-1} + \mathcal{C}P_{n-2},\\ \mathcal{C}P_{n+2} &= \mathcal{C}P_n + \mathcal{C}P_{n-1},\\ \mathcal{C}P_{n+3} &= \mathcal{C}P_{n+1} + \mathcal{C}P_n\end{aligned}$$

If we sum of both of sides of the identities above, we obtain,

$$\mathcal{C}P_{n+3}+\mathcal{C}P_{n+2}=\mathcal{C}P_1+\mathcal{C}P_2+\sum_{i=0}^{n}\mathcal{C}P_i.$$

Hence, we get the desired result.

Theorem 10. *The partial sum of the first* n *terms of the bicomplex Perrin sequence* $\{\mathcal{C}R_n\}$ *is*

$$\sum_{i=0}^{n}\mathcal{C}R_i=\mathcal{C}R_{n+5}-2-5i-5j-7ij, \quad n\geq 0.$$

Proof. We know that

$$\mathcal{C}R_{n+3}=\mathcal{C}R_{n+1}+\mathcal{C}R_n$$

So, applying to the identity above, we deduce that

$$\begin{aligned}\mathcal{C}R_3&=\mathcal{C}R_1+\mathcal{C}R_0,\\ \mathcal{C}R_4&=\mathcal{C}R_2+\mathcal{C}R_1,\\ \mathcal{C}R_5&=\mathcal{C}R_3+\mathcal{C}R_2,\\ &\ldots,\\ \mathcal{C}R_{n+1}&=\mathcal{C}R_{n-1}+\mathcal{C}R_{n-2},\\ \mathcal{C}R_{n+2}&=\mathcal{C}R_n+\mathcal{C}R_{n-1},\\ \mathcal{C}R_{n+3}&=\mathcal{C}R_{n+1}+\mathcal{C}R_n\end{aligned}$$

If we sum of both of sides of the identities above, we obtain,

$$\mathcal{C}R_{n+3}+\mathcal{C}R_{n+2}=\mathcal{C}R_1+\mathcal{C}R_2+\sum_{i=0}^{n}\mathcal{C}R_i.$$

Hence, we get the desired result.

Theorem 11. *Let* m *be a positive integer. Then,*

$$\sum_{n=0}^{m}\binom{m}{n}\mathcal{C}P_n=\mathcal{C}P_{3m}$$

Proof. Applying Binet-like formula (8), we obtain the identities

$$\begin{aligned}\sum_{n=0}^{m}\binom{m}{n}\mathcal{C}P_n &= \sum_{n=0}^{m}\binom{m}{n}(a\hat{\alpha}\alpha^n + b\hat{\beta}\beta^n + c\hat{\gamma}\gamma^n)\\ &= \sum_{n=0}^{m}\binom{m}{n}(a\hat{\alpha}(\alpha)^n1^{m-n} + b\hat{\beta}(\beta)^n1^{m-n} + c\hat{\gamma}(\gamma)^n1^{m-n})\end{aligned}$$

Note that, for any real numbers a and b, and any positive integer m, the identity

$$(a+b)^m = \sum_{n=0}^{m}\binom{m}{n}a^nb^{m-n} \tag{10}$$

holds. Hence

$$a\hat{\alpha}(\alpha+1)^m + b\hat{\beta}(\beta+1)^m + c\hat{\gamma}(\gamma+1)^m$$

$\alpha^3 = \alpha+1$, $\beta^3 = \beta+1$ and $\gamma^3 = \gamma+1$ are due to (5). Hence,

$$a\hat{\alpha}\alpha^{3m} + b\hat{\beta}\beta^{3m} + c\hat{\gamma}\gamma^{3m}$$

Thus, the proof is completed.

Theorem 12. *Let m be a positive integer. Then,*

$$\sum_{n=0}^{m}\binom{m}{n}\mathcal{C}R_n = \mathcal{C}R_{3m}$$

Proof. Applying Binet-like formula (9) and combining this with (10) and (5) we obtain the identity

$$\begin{aligned}\sum_{n=0}^{m}\binom{m}{n}\mathcal{C}R_n &= \sum_{n=0}^{m}\binom{m}{n}(\hat{\alpha}\alpha^n + \hat{\beta}\beta^n + \hat{\gamma}\gamma^n)\\ &= \sum_{n=0}^{m}\binom{m}{n}(\hat{\alpha}(\alpha)^n1^{m-n} + \hat{\beta}(\beta)^n1^{m-n} + \hat{\gamma}(\gamma)^n1^{m-n})\\ &= \hat{\alpha}(\alpha+1)^m + \hat{\beta}(\beta+1)^m + \hat{\gamma}(\gamma+1)^m\\ &= \hat{\alpha}\alpha^{3m} + \hat{\beta}\beta^{3m} + \hat{\gamma}\gamma^{3m}\end{aligned}$$

Thus, the proof is completed.

Theorem 13. *Let m be a positive integer. Then,*

$$\sum_{k=0}^{m}\binom{m}{k}\mathcal{C}P_{n-k} = \mathcal{C}P_{n+2m}$$

Proof. Applying Binet-like formula (8) and combining this with (10) and (5) we obtain the identity

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\mathcal{C}P_{n-k} &= \sum_{k=0}^{m}\binom{m}{k}(a\hat{\alpha}\alpha^{n-k}+b\hat{\beta}\beta^{n-k}+c\hat{\gamma}\gamma^{n-k}) \\ &= \sum_{k=0}^{m}\binom{m}{k}(a\hat{\alpha}(\alpha)^{m-k}1^{k}\alpha^{n-m}+b\hat{\beta}(\beta)^{m-k}1^{k}\beta^{n-m} \\ &\quad +c\hat{\gamma}(\gamma)^{m-k}1^{k}\gamma^{n-m}) \\ &= a\hat{\alpha}(\alpha+1)^{m}\alpha^{n-m}+b\hat{\beta}(\beta+1)^{m}\beta^{n-m}+c\hat{\gamma}(\gamma+1)^{m}\gamma^{n-m} \\ &= a\hat{\alpha}\alpha^{n+2m}+b\hat{\beta}\beta^{n+2m}+c\hat{\gamma}\gamma^{n+2m}\end{aligned}$$

Thus, the proof is completed.

Theorem 14. *Let m be a positive integer. Then,*

$$\sum_{k=0}^{m}\binom{m}{k}\mathcal{C}R_{n-k}=\mathcal{C}R_{n+2m}$$

Proof. Applying Binet-like formula (9) and combining this with (10) and (5) we obtain the identity

$$\begin{aligned}\sum_{k=0}^{m}\binom{m}{k}\mathcal{C}R_{n-k} &= \sum_{k=0}^{m}\binom{m}{k}(\hat{\alpha}\alpha^{n-k}+\hat{\beta}\beta^{n-k}+\hat{\gamma}\gamma^{n-k}) \\ &= \sum_{k=0}^{m}\binom{m}{k}(\hat{\alpha}(\alpha)^{m-k}1^{k}\alpha^{n-m}+\hat{\beta}(\beta)^{m-k}1^{k}\beta^{n-m} \\ &\quad +\hat{\gamma}(\gamma)^{m-k}1^{k}\gamma^{n-m}) \\ &= \hat{\alpha}(\alpha+1)^{m}\alpha^{n-m}+\hat{\beta}(\beta+1)^{m}\beta^{n-m}+\hat{\gamma}(\gamma+1)^{m}\gamma^{n-m} \\ &= \hat{\alpha}\alpha^{n+2m}+\hat{\beta}\beta^{n+2m}+\hat{\gamma}\gamma^{n+2m}\end{aligned}$$

Thus, the proof is completed.

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