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A NEW FAMILY OF HARMONIC FUNCTIONS DEFINED BY AN INTEGRAL OPERATOR

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Abstract. In this paper we define and investigate new classes of harmonic multivalent functions denoted by $HS_m(\lambda, p, \alpha)$ and $HS_m^*(\lambda, p, \alpha)$. These classes are defined by making use of a modified Sălăgean integral operator. We obtained coefficient inequalities results and distortion bounds relations for the functions in the new classes. We also determine the extreme points of closed convex hulls of $HS_m^*(\lambda, p, \alpha)$ denoted by $clcoHS_m^*(\lambda, p, \alpha)$.

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1. Introduction

Let $\mathcal{H}$ denote the family of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the unit disc $\Delta = \{z : |z| < 1\}$ with the normalization

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1}$$

The class $\mathcal{H}$ was introduced by Clunie and T. Sheil-Small [4]. Also see an excellent monograph written by Duren [5]. Recent interest in the study of multivalent harmonic functions in the plane prompted the publication of several articles, such as [3], [1], [6].

Recently Ahuja and Jahangiri [2] defined the class $\mathcal{H}_p(n), p, n \in \mathbb{N} = \{1, 2, 3, \dots, \}$ consisting of all $p-$valent harmonic functions $f = h + \bar{g}$ that are sense preserving in Δ and h and g are of the form

$$h(z) = z^p + \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1}, \quad g(z) = \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1}, \quad |b_p| < 1. \tag{2}$$

The integral operator I^m was introduced by Sălăgean [7].

For $f = h + \bar{g}$ given by (1) Ahuja and Jahangiri [2] introduced the modified Sălăgean operator for harmonic functions. Te modified Sălăgean operator, for multivalent harmonic functions $f = h + \bar{g}$ given by (2), has been introduced in [2] and is written in the form

$$I^m f(z) = I^m h(z) + (-1)^m \overline{I^m g(z)},\ p > m \tag{3}$$

where

$$I^m h(z) = z^p + \sum_{k=2}^{\infty} \left(\frac{p}{k+p-1}\right)^m a_{k+p-1} z^{k+p-1}$$

and

$$I^m g(z) = \sum_{k=1}^{\infty} \left(\frac{p}{k+p-1}\right)^m b_{k+p-1} z^{k+p-1}.$$

The integral operator I^m is defined in [7] by:

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = I f(z) = p \int_0^z f(t) t^{-1} dt,$$

$$\ldots$$

$$I^m f(z) = I\left(I^{m-1} f(z)\right),\ m \in \mathbb{N},\ f \in \mathcal{A}$$

where $\mathcal{A} = \left\{ f \in \mathcal{H}\colon\ f(z) = z + \sum_{k=2}^{\infty} a_k z^k \right\}$ and $\mathcal{H} = \mathcal{H}(U)$.

2. Main results

Definition 1. *Consider $0 \le \alpha < 1$, $m \in \mathbb{N}_0$, $\lambda > \frac{1}{2}$ and $z \in \Delta$. A function $f = h + \bar{g}$ with h and $\bar{g}$ given by (1) belongs to class $HS_m(\lambda, p, \alpha)$ if*

$$\operatorname{Re}\left\{ \frac{I^{m+1} f(z) + \lambda\left(I^m f(z) - I^{m+1} f(z)\right)}{z^p} \right\} > \alpha \tag{4}$$

where I^m is defined by (3).

Definition 2. *Let $0 \le \alpha < 1$, $m \in \mathbb{N}_0$, $\lambda > \frac{1}{2}$ and the complex number $z \in \Delta$. We say that a harmonic function $f_m = h + \bar{g}_m$ belongs to class $HS^*_m(\lambda, p, \alpha)$ if $f_m \in HS_m(\lambda, p, \alpha)$ and the functions h and g_m are of the form*

$$h(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1},\quad g_m(z) = (-1)^{m-1} \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1},\quad |b_p| < 1 \tag{5}$$

where $a_{k+p-1}, b_{k+p-1} \ge 0, |b_p| < 1$.

We obtain coefficient inequality results which gives sufficient conditions for a function f to be in the class $HS_m(\lambda,p,\alpha)$. This coefficient inequality result is indeed a necessary condition for f to be in the $HS_m^*(\lambda,p,\alpha)$. Hence, for this inequality, we deduce extreme points results, convolution conditions, distortion bounds, and convex combinations for the new class $HS_m^*(\lambda,p,\alpha)$.

We begin with two sufficient conditions for functions in $HS_m(\lambda,p,\alpha)$.

Theorem 1. *Let $f=h+\bar{g}$ such that h and g are given by (2). If*

$$\sum_{k=1}^{\infty}\left(\left[\lambda\left(\frac{p}{k+p-1}\right)^m+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]|a_{k+p-1}|\right.$$
$$\left.+\left[\lambda\left(\frac{p}{k+p-1}\right)^m-(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]|b_{k+p-1}|\right)\leq 2-\alpha, \tag{6}$$

$a_p=1, 0\leq\alpha<1, m\in\mathbb{N}_0, \lambda>\frac{1}{2}$, then we deduce for f a harmonic univalent property, sense preserving in Δ and $f\in HS_m(\lambda,p,\alpha)$.

Proof. If $z_1\neq z_2$ then

$$\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right|$$

$$=1-\left|\frac{\sum_{k=1}^{\infty}b_{k+p-1}\left(z_1^{k+p-1}-z_2^{k+p-1}\right)}{(z_1-z_2)\sum_{k=2}^{\infty}a_{k+p-1}\left(z_1^{k+p-1}-z_2^{k+p-1}\right)}\right|>1-\frac{\sum_{k=1}^{\infty}(k+p-1)|b_{k+p-1}|}{1-\sum_{k=2}^{\infty}(k+p-1)|a_{k+p-1}|}$$

$$\geq 1-\frac{\sum_{k=1}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^m-(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]|b_{k+p-1}|}{\sum_{k=2}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^m+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]|a_{k+p-1}|}\geq 0,$$

which prove univalence.

Note that f is sense preserving in Δ. This is because

$$|h'(z)|\geq 1-\sum_{k=2}^{\infty}(k+p-1)|a_{k+p-1}||z|^{k+p-2}$$

$$>1-\sum_{k=2}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^m+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]|a_{k+p-1}|$$

$$
\begin{aligned}
&> \sum_{k=1}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}-(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]\left|b_{k+p-1}\right| \\
&\geq \sum_{k=1}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}-(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]\left|b_{k+p-1}\right||z|^{k+p-2} \\
&\geq \sum_{k=1}^{\infty}\left(_{k+p-1}\right)\left|b_{k+p-1}\right||z|^{k+p-2} \geq \left|g'(z)\right| .
\end{aligned}
$$

Using the fact that Re $\{w\} \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$ is suffices to show that

$$
\begin{aligned}
&\left|(1-\alpha) z^{p}+I^{m+1} f(z)+\lambda\left(I^{m} f(z)+I^{m+1} f(z)\right)\right| \\
&-\left|(1+\alpha) z^{p}-I^{m+1} f(z)-\lambda\left(I^{m} f(z)-I^{m+1} f(z)\right)\right| \\
&= \left|(2-\alpha) z^{p}+\sum_{k=2}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right] a_{k+p-1} z^{k+p-1}\right. \\
&\left.-(-1)^{m+1} \sum_{k=1}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}-(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right] \overline{b_{k+p-1} z^{k+p-1}}\right| \\
&-\left|\alpha z^{p}-\sum_{k=2}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right] a_{k+p-1} z^{k+p-1}\right. \\
&\left.+(-1)^{m+1} \sum_{k=1}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}-(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right] \overline{b_{k+p-1} z^{k+p-1}}\right| \\
&= \left|(2-\alpha) z^{p}+\sum_{k=2}^{\infty} 2\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right] a_{k+p-1} z^{k+p-1}\right. \\
&\left.-(-1)^{m+1} \sum_{k=1}^{\infty} 2\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}-(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right] \overline{b_{k+p-1} z^{k+p-1}}\right| \\
&\geq (2-\alpha)|z|^{p}-\sum_{k=2}^{\infty} 2\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]\left|a_{k+p-1}\right||z|^{k+p-1} \\
&-(-1)^{m} \sum_{k=1}^{\infty} 2\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}-(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]\left|\overline{b}_{k+p-1}\right||\overline{z}|^{k+p-1}
\end{aligned}
$$

$$> 2\left\{(1-\alpha)-\left(\sum_{k=1}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]|a_{k+p-1}|\right.\right.$$
$$\left.\left.+\sum_{k=1}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}-(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]|b_{k+p-1}|\right)\right\}>0.$$

So, the proof of theorem is complete.

The harmonic univalent function

$$f(z)=z^{p}+\sum_{k=2}^{\infty}\frac{1-\alpha}{\lambda\left(\frac{p}{k+p-1}\right)^{m}+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}\cdot x_{k}z^{k+p-1} \tag{7}$$
$$+\sum_{k=1}^{\infty}\frac{1-\alpha}{\lambda\left(\frac{p}{k+p-1}\right)^{m}+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}\cdot\overline{y_{k}z^{k+p-1}},$$

$0\leq\alpha<1, m\in\mathbb{N}_0, \lambda>\frac{1}{2}$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, show that sufficient bound given by (6) is sharp. The function of the form (7) are in the class $HS_m(\lambda,p,\alpha)$ because

$$\sum_{k=1}^{\infty}\left\{\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]|a_{k+p-1}|\right.$$
$$\left.+\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}-(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]|b_{k+p-1}|\right\}$$
$$=1+(1-\alpha)\left(\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|\right)=2-\alpha.$$

In the following theorem, it is shown that the condition (6) is also necessary for function $f_m=h+\bar{g}_m$, where h and g_m are of the form (5).

Theorem 2. *Let $f_m=h+\bar{g}_m$ be given by (5). Then $f_m\in HS_m^*(\lambda,p,\alpha)$ if and only if*

$$\sum_{k=1}^{\infty}\left\{\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]a_{k+p-1}\right.$$
$$\left.+\left[\lambda\left(\frac{p}{k+p-1}\right)^{m}-(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]b_{k+p-1}\right\}\leq 2-\alpha. \tag{8}$$

Proof. Since $HS_m^*(\lambda,p,\alpha)\subset HS_m(\lambda,p,\alpha)$ we need to proof the "only if" part of the theorem. To this end for function f_m of the form (5) we notice that the condition

$$\mathrm{Re}\left\{\frac{I^{m+1}f(z)+\lambda\left(I^m f(z)-I^{m+1}f(z)\right)}{z^p}\right\}\geq\alpha$$

is equivalent to

$$\mathrm{Re}\left\{\begin{array}{c}(1-\alpha)-\sum_{k=2}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^m+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]a_{k+p-1}z^{k-1}-\\ -(-1)^{m+1}\sum_{k=1}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^m-(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]\overline{b_{k+p-1}z^{k-1}}\end{array}\right\}\geq 0. \tag{9}$$

If we choose z to be real and let $z\to 1^-$, we get

$$(1-\alpha)-\sum_{k=2}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^m+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]a_{k+p-1}$$

$$-(-1)^{m+1}\sum_{k=1}^{\infty}\left[\lambda\left(\frac{p}{k+p-1}\right)^m-(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}\right]\overline{b_{k+p-1}}\geq 0 \tag{10}$$

which is precisely the assertion (8).

Next, we determine the extreme points of closed convex hull of $HS_m^*(\lambda,p,\alpha)$ denoted by $clcoHS_m^*(\lambda,p,\alpha)$.

Theorem 3. *Let $f_m=h+\bar{g}_m$ be given by (5). Then $f_m\in HS_m^*(\lambda,p,\alpha)$ if and only if*

$$f_m(z)=\sum_{k=1}^{\infty}x_{k+p-1}h_{k+p-1}(z)+y_{k+p-1}g_{m_{k+p-1}}(z) \tag{11}$$

where $h_p(z)=z^p$,

$$h_{k+p-1}(z)=z^p-\frac{1-\alpha}{\lambda\left(\frac{p}{k+p-1}\right)^m+(1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}\overline{z}^{k+p-1},$$

$k=1,2,3,\ldots,m\in\mathbb{N}_0, x_{k+p-1}>0, y_{k+p-1}\geq 0, x_p=1-\sum_{k=2}^{\infty}x_{k+p-1}-\sum_{k=1}^{\infty}y_{k+p-1}.$

The extreme points of $HS_m^*(\lambda,p,\alpha)$ are $\{h_{k+p-1}\}$ and $\{g_{m_{k+p-1}}\}$.

Proof. For function f_m of the form (11) we have

$$f_m(z) = \sum_{k=1}^{\infty} x_{k+p-1} h_{k+p-1}(z) + y_{k+p-1} g_{m_{k+p-1}}(z)$$

$$= \sum_{k=1}^{\infty} (x_{k+p-1} + y_{k+p-1}) z^p - \sum_{k=1}^{\infty} \frac{1-\alpha}{\lambda \left(\frac{p}{k+p-1}\right)^m + (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}} \cdot x_{k+p-1} z^{k+p-1}$$

$$+ (-1)^{m-1} \sum_{k=1}^{\infty} \frac{1-\alpha}{\lambda \left(\frac{p}{k+p-1}\right)^m + (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}} \cdot y_{k+p-1} \overline{z}^{k+p-1}.$$

Then

$$\sum_{k=2}^{\infty} \frac{\lambda \left(\frac{p}{k+p-1}\right)^m + (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{1-\alpha} \cdot a_{k+p-1}$$

$$+ \sum_{k=2}^{\infty} \frac{\lambda \left(\frac{p}{k+p-1}\right)^m + (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{1-\alpha} \cdot b_{k+p-1}$$

$$= \sum_{k=2}^{\infty} x_{k+p-1} + \sum_{k=1}^{\infty} y_{k+p-1} = 1 - x_p \leq 1$$

and so $f_m \in clcoH(\lambda, p, \alpha)$.

Conversely, suppose that $f_m \in HS_m^*(\lambda, p, \alpha)$. Setting

$$x_p = 1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1},$$

$$x_{k+p-1} = \frac{\lambda \left(\frac{p}{k+p-1}\right)^m + (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{1-\alpha} \cdot a_{k+p-1}, k = 1, 2, 3, \ldots, m \in \mathbb{N}_0$$

and

$$y_{k+p-1} = \frac{\lambda \left(\frac{p}{k+p-1}\right)^m - (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{1-\alpha} \cdot b_{k+p-1}, k = 1, 2, 3, \ldots, m \in \mathbb{N}_0.$$

We obtain the required representative since

$$f_m(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} b_{k+p-1} \overline{z}^{k+p-1}$$

$$= z^p - \sum_{k=2}^{\infty} \frac{1-\alpha}{\lambda\left(\frac{p}{k+p-1}\right)^m + (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}} x_{k+p-1} z^{k+p-1}$$

$$+ (-1)^{m-1} \sum_{k=1}^{\infty} \frac{1-\alpha}{\lambda\left(\frac{p}{k+p-1}\right)^m - (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}} y_{k+p-1} \overline{z}^{k+p-1}$$

$$= z^p - \sum_{k=2}^{\infty} \left[z^p - h_{k+p-1}(z)\right] x_{k+p-1} - \sum_{k=1}^{\infty} \left[z^p - g_{m_{k+p-1}}(z)\right] y_{k+p-1}$$

$$= \left[1 - \sum_{k=2}^{\infty} x_{k+p-1} - \sum_{k=1}^{\infty} y_{k+p-1}\right] z^p + \sum_{k=2}^{\infty} x_{k+p-1} h_{k+p-1}(z) + \sum_{k=1}^{\infty} y_{k+p-1} g_{m_{k+p-1}}(z)$$

$$= \sum_{k=1}^{\infty} x_{k+p-1} h_{k+p-1}(z) + y_{k+p-1} g_{m_{k+p-1}}(z).$$

The following theorem gives the distortion bounds results for functions in $HS_m^*(\lambda, p, \alpha)$ which produce a covering result.

Theorem 4. *Let* $f_m = h + \bar{g}$ *be given by* (5) *and* $f_m \in HS_m^*(\lambda, p, \alpha)$*. Then for* $|z| = r < 1$ *we have*

$$|f_m(z)| \le (1 + |b_p|) r^p + \frac{1}{\lambda\left(\frac{p}{p+1}\right)^m + (1-\lambda)\left(\frac{p}{p+1}\right)^{m+1}} \left[(1-\alpha) - (2\lambda - 1)|b_p|\right] r^{p+1}$$

and

$$|f_m(z)| \ge (1 - |b_p|) r^p - \frac{1}{\lambda\left(\frac{p}{p+1}\right)^m + (1-\lambda)\left(\frac{p}{p+1}\right)^{m+1}} \left[(1-\alpha) - (2\lambda - 1)|b_p|\right] r^{p+1}.$$

Proof. Let $f_m \in HS_m^*(\lambda, p, \alpha)$. Taking the absolute value of f_m we obtain

$$|f_m(z)| = \left| z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} b_{k+p-1} z^{k+p-1} \right|$$

$$\le r^p + \sum_{k=2}^{\infty} |a_{k+p-1}| r^{k+p-1} + \sum_{k=2}^{\infty} |b_{k+p-1}| r^{k+p-1}$$

$$\le r^p + |b_p| r^p + \sum_{k=2}^{\infty} \left(|a_{k+p-1}| + |b_{k+p-1}|\right) r^{p+1}$$

$$= (1 + |b_p|)\, r^p + \frac{1}{\lambda \left(\frac{p}{p+1}\right)^m + (1-\lambda)\left(\frac{p}{p+1}\right)^{m+1}} \cdot$$

$$\cdot \sum_{k=2}^{\infty} \left[\lambda \left(\frac{p}{p+1}\right)^m + (1-\lambda)\left(\frac{p}{p+1}\right)^{m+1} \right] (|a_{k+p-1}| + |b_{k+p-1}|)$$

$$\leq (1 + |b_p|)\, r^p + \frac{1}{\lambda \left(\frac{p}{p+1}\right)^m + (1-\lambda)\left(\frac{p}{p+1}\right)^{m+1}} \cdot$$

$$\cdot \sum_{k=2}^{\infty} \left\{ \left[\lambda \left(\frac{p}{k+p-1}\right)^m + (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1} \right] |a_{k+p-1}| \right.$$

$$\left. + \left[\lambda \left(\frac{p}{k+p-1}\right)^m - (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1} \right] |b_{k+p-1}| \right\} r^{p+1}$$

$$\leq (1 + |b_p|)\, r^p + \frac{1}{\lambda \left(\frac{p}{p+1}\right)^m + (1-\lambda)\left(\frac{p}{p+1}\right)^{m+1}} \left[(1-\alpha) - (2\lambda - 1)\,|b_p|\right] r^{p+1}.$$

Similarly, we can prove the left hand inequality.

For our next theorem, we need to recall the well known notion convolution of two multivalent functions. For the functions

$$f_m(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} b_{k+p-1} \overline{z}^{k+p-1}$$

and

$$F_m(z) = z^p - \sum_{k=2}^{\infty} A_{k+p-1} z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} B_{k+p-1} \overline{z}^{k+p-1}$$

where $A_{k+p-1}, B_{k+p-1} \geq 0$,

$$(f_m * F_m)(z) = f_m(z) * F_m(z)$$

$$= z^p - \sum_{k=2}^{\infty} a_{k+p-1} A_{k+p-1} z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} b_{k+p-1} B_{k+p-1} \overline{z}^{k+p-1}. \tag{12}$$

Theorem 5. *For $0 \leq \beta \leq \alpha < 1$, let $f_m \in HS_m^*(\lambda, p, \alpha)$ and $F_m \in HS_m^*(\lambda, p, \beta)$. Then*

$$f_m * F_m \in HS_m^*(\lambda, p, \alpha) \subset H_m^*(\lambda, p, \beta).$$

Proof. For f_m and F_m as in Theorem 6 we can write

$$f_m(z) = z^p - \sum_{k=2}^{\infty} a_{k+p-1} z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} b_{k+p-1} \overline{z}^{k+p-1}$$

and

$$F_m(z) = z^p - \sum_{k=2}^{\infty} A_{k+p-1} z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} B_{k+p-1} \overline{z}^{k+p-1}.$$

Then the convolution $f_m * F_m$ is given by (12) . We wish to show that the coefficients of $f_m * F_m$ satisfy the required condition given by Theorem 2.

For $F_m \in HS_m^*(\lambda, p, \beta)$ we note that $|A_{k+p-1}| \leq 1$ and $|B_{k+p-1}| \leq 1$. Now, for convolution functions $f_m * F_m$ we obtain

$$\sum_{k=2}^{\infty} \frac{\lambda\left(\frac{p}{k+p-1}\right)^m + (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{2-\beta} a_{k+p-1} A_{k+p-1}$$

$$+ \sum_{k=1}^{\infty} \frac{\lambda\left(\frac{p}{k+p-1}\right)^m - (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{2-\beta} b_{k+p-1} B_{k+p-1}$$

$$\leq \sum_{k=2}^{\infty} \frac{\lambda\left(\frac{p}{k+p-1}\right)^m + (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{2-\beta} a_{k+p-1}$$

$$+ \sum_{k=1}^{\infty} \frac{\lambda\left(\frac{p}{k+p-1}\right)^m - (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{2-\beta} b_{k+p-1}$$

$$\leq \sum_{k=2}^{\infty} \frac{\lambda\left(\frac{p}{k+p-1}\right)^m + (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{2-\alpha} a_{k+p-1}$$

$$+ \sum_{k=1}^{\infty} \frac{\lambda\left(\frac{p}{k+p-1}\right)^m - (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{2-\alpha} b_{k+p-1},$$

since $0 \leq \beta \leq \alpha < 1$ and $f_m \in H(\lambda, p, \alpha)$. Therefore

$$f_m * F_m \in HS_m^*(\lambda, p, \alpha) \subset HS_m^*(\lambda, p, \beta).$$

Now, we show that $HS_m^*(\lambda, p, \alpha)$ is closed under convex combination.

Theorem 6. *The family $HS_m^*(\lambda, p, \alpha)$ is closed under convex combination.*

Proof. For $i = 1, 2, \ldots$. Suppose that $f_{m_i} \in HS_m^*(\lambda, p, \alpha)$, where

$$f_{m_i}(z) = z^p - \sum_{k=2}^{\infty} a_{i_{k+p-1}} z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} b_{i_{k+p-1}} \overline{z}^{k+p-1}$$

where $a_{i_{k+p-1}}, b_{i_{k+p-1}} \geq 0$. Then by Theorem 2

$$\sum_{k=2}^{\infty} \frac{\lambda\left(\frac{p}{k+p-1}\right)^m + (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{2-\beta} a_{i_{k+p-1}} \tag{13}$$

$$+ \sum_{k=1}^{\infty} \frac{\lambda\left(\frac{p}{k+p-1}\right)^m - (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{2-\beta} b_{i_{k+p-1}} \leq 1. \tag{14}$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_{m_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{m_i}(z) = z^p - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{i_{k+p-1}} \right) z^{k+p-1} + (-1)^{m-1} \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i b_{i_{k+p-1}} \right) \overline{z}^{k+p-1}.$$

Then by (14)

$$\sum_{k=1}^{\infty} \frac{\lambda\left(\frac{p}{k+p-1}\right)^m + (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{2-\alpha} \left(\sum_{i=1}^{\infty} t_i a_{i_{k+p-1}} \right)$$

$$+ \sum_{k=1}^{\infty} \frac{\lambda\left(\frac{p}{k+p-1}\right)^m - (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{2-\alpha} \left(\sum_{i=1}^{\infty} t_i b_{i_{k+p-1}} \right)$$

$$= \sum_{i=1}^{\infty} t_i \left(\sum_{k=2}^{\infty} \frac{\lambda\left(\frac{p}{k+p-1}\right)^m + (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{2-\alpha} a_{i_{k+p-1}} \right.$$

$$\left. + \sum_{k=1}^{\infty} \frac{\lambda\left(\frac{p}{k+p-1}\right)^m - (1-\lambda)\left(\frac{p}{k+p-1}\right)^{m+1}}{2-\alpha} b_{i_{k+p-1}} \right) \leq \sum_{i=1}^{\infty} t_i = 1$$

and therefore $\sum_{i=1}^{\infty} t_i f_{m_i} \in HS_m^*(\lambda, p, \alpha)$.

3. Conclusions

In the present paper we have proposed two classes of harmonic multivalent functions. They are denoted by $HS_m(\lambda, p, \alpha)$ and $HS_m^*(\lambda, p, \alpha)$. These classes are defined by using a modified Sălăgean integral operator. We succeeded to proof certain properties for the above mentioned classes. We also deduce the extreme points of closed convex hulls of $HS_m^*(\lambda, p, \alpha)$ denoted by $clcoHS_m^*(\lambda, p, \alpha)$. In the final part we consider the convolution of two multivalent functions in order to establish a property related to the class $HS_m^*(\lambda, p, \alpha)$ namely, we deduce that $HS_m^*(\lambda, p, \alpha)$ is closed under convex combination.

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