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THE CYCLIC CODES OF ODD LENGTH OVER THE FINITE RING $R_{u^4,v^2,2}$ AND THEIRS DNA APPLICATIONS

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Abstract. Necessary and sufficient conditions for cyclic codes of odd length over the finite ring $R_{u^4,v^2,2} = F_2[u,v]/ < u^4, v^2, uv - vu > \cong F_2 + uF_2 + u^2F_2 + u^3F_2 + vF_2 + vuF_2 + vu^2F_2 + vu^3F_2$, where $u^4 = 0, v^2 = 0, uv = vu$ and $F_2 = \{0,1\}$ to have the DNA properties are determined. A correspondence between the elements of the finite ring $R_{u^4,v^2,2}$ and DNA 4-bases is given. By using the correspondence and the cyclic DNA codes over the finite ring $R_{u^4,v^2,2}$, the DNA codes are obtained.

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1. Introduction

DNA has a sophisticated structure with an perfect error correcting property. Some scientists used some special error correcting codes that enjoy some properties with DNA structure in order to understand DNA. To model DNA, they considered some special error correcting codes over many finite rings or finite fields with 4^k elements. One of them is cyclic DNA code. A cyclic DNA code means a cyclic code that has reversible complement property. By using a correspondence from the finite rings or finite fields to DNA bases, they obtained DNA codes via cyclic DNA codes.

A DNA code of length n means a set of codewords $(a_1, ..., a_n)$ where $a_i \in \{A, T, G, C\}$. Designing DNA codes is very important issue especially in DNA computing.

In [8], B. Yildiz, I. Siap gave a sufficient condition for a cyclic code of odd length over $F_2[u]/ < u^4 - 1 >$ to be reversible complement. They constructed DNA codes as images of reversible complement cyclic codes of odd length n over $F_2[u]/ < u^4 - 1 >$. In [3], S. Bathala and M. Bhaintwal gave a sufficient condition for a cyclic code of odd length over $F_2[u,v]/ < u^2, v^2, uv - vu >$ to be reversible complement. They constructed DNA codes as images of reversible complement cyclic codes of odd length n over $F_2[u,v]/ < u^2, v^2, uv - vu >$.

Motivated from these works, we are interested in cyclic DNA codes over $R_{u^4,v^2,2}$ in order to construct DNA codes via them.

This paper is organised as follows. In section 2, some knowledges and some notations about the finite ring $R_{u^4,v^2,2}$ are given. In section 3, the structures of cyclic codes over this ring are given. In section 4, the necessary and sufficient conditions for cyclic codes of odd length over this ring to be reversible are determined. In section 5, a correspondence between the elements of this ring and DNA 4-bases is given. In section 6, the necessary and sufficient conditions for cyclic codes of odd length over this ring to be reversible complement are given and DNA codes are obtained via cyclic DNA codes of odd length n over this ring.

2. The ring $R_{u^4,v^2,2}$

In [4], the finite ring $R_{u^k,v^2,p}$, where k is a positive integer, p is a prime number is introduced. By taking $p=2, k=4$, it is said that the finite ring $R_{u^4,v^2,2} = F_2[u,v]/ < u^4, v^2, uv-vu > \cong F_2 + uF_2 + u^2F_2 + u^3F_2 + vF_2 + vuF_2 + vu^2F_2 + vu^3F_2$ is local with the unique maximal ideal $\langle u, v\rangle$, where $u^4=0, v^2=0, uv=vu$, and $F_2=\{0,1\}$. The set $\{\{0\}, < u >, < u^2 >, < u^3 >, < uv >, < u^2v >, < u^3v >, < u+v >, < u^2+v >, < u^3+v >, < u^3, v >, < u^2, v >, < u, v >, < 1 >\}$ gives list of all ideals of $R_{u^4,v^2,2}$. It is also commutative with characteristic 2 and 256 elements.

It can be viewed as

$$R_{u^4,v^2,2} = R_{u^4,2} + vR_{u^4,2}, \quad v^2=0$$

where $R_{u^4,2} = F_2[u]/ < u^4 >$.

By using the following two maps,

$$\begin{array}{rcl} \phi & : & R_{u^4,v^2,2} \longrightarrow \left(R_{u^4,2}\right)^2 \\ x+vy & \longmapsto & (x, x+y) \end{array}$$

where $x, y \in R_{u^4,2}$.

$$\begin{array}{rcl} \eta & : & R_{u^4,2} \longrightarrow \left(F_2\right)^4 \\ a+ub+cu^2+du^3 & \longmapsto & (a+b+c+d, c+d, b+d, d) \end{array}$$

where $a, b, c, d \in F_2$ which was defined in [7], we can define a new Gray map from $R_{u^4,v^2,2}$ to F_2^8 as follows,

$$\psi : R_{u^4,v^2,2} \longrightarrow F_2^8$$
$$\alpha \longmapsto (\eta(\phi(\alpha))$$

where $\psi = \eta\phi$.

A linear code C of length n over the finite ring R (or finite field F_q) is $R-$submodule (F_q- subspace) of $R^n(F_q^n)$.

Let D be a linear code over F_2 of length m and $\acute{c} = (\acute{c}_0, \acute{c}_1, ..., \acute{c}_{m-1})$ be a codeword of D. The Hamming weight of $\acute{c}$ is defined as $w_H(\acute{c}) = \sum_{i=0}^{m-1} w_H(\acute{c}_i)$ where $w_H(\acute{c}_i) = 1$ if $\acute{c}_i \neq 0$ and $w_H(\acute{c}_i) = 0$ if $\acute{c}_i = 0$. Minimum Hamming distance of D is defined as $d_H = d_H(D) = \min d_H(c, \acute{c})$, where for any $\acute{c} \in D$, $c \neq \acute{c}$ and $d_H(c, \acute{c})$ is Hamming distance between two codewords with $d_H(c, \acute{c}) = w_H(c - \acute{c})$.

The Gray weight of the element α in $R_{u^4,v^2,2}$ is defined

$$w_G(\alpha) = w_H(\psi(\alpha))$$

for all $\alpha \in R_{u^4,v^2,2}$.

Let C be a linear code over $R_{u^4,v^2,2}$ of length n. For any codeword $c = (c_0, ..., c_{n-1})$, the Gray weight of c is defined as $w_G(c) = \sum_{i=0}^{n-1} w_G(c_i)$ and the minimum Gray distance of C is defined as $d_G = d_G(C) = \min d_G(c, c')$, where for any $c' \in C, c \neq c'$ and $d_G(c, c')$ is Gray distance between two codewords with $d_G(c, c') = w_G(c - c')$.

The minimum Gray weight of C is the smallest nonzero Gray weight among all codewords. If C is linear, the minimum Gray distance is the same as the minimum Gray weight.

The Gray map is extended to $R^n_{u^4,v^2,2}$ naturally as follows

$$\psi : \left(R^n_{u^4,v^2,2}, \text{Gray weight}\right) \longrightarrow \left(F_2^{8n}, \text{Hamming weight}\right)$$

$$(c_0, c_1, ..., c_{n-1}) \longmapsto (\psi(c_0), ..., \psi(c_{n-1}))$$

for all $c_i \in R_{u^4,v^2,2}$, where $i = 0, ..., n-1$.

Since the new Gray map is linear and distance preserving, the following theorem is obtained

Theorem 1. *If C is a linear code over $R_{u^4,v^2,2}$ of length n size M, and minimum Gray weight d_G, then $\psi(C)$ is a binary linear code with parameters $(8n, M, d_G = d_H)$.*

C is called cyclic code if it is closed with respect to cyclic shift, i.e. $(c_{n-1}, c_0, ..., c_{n-2}) \in C$ whenever $(c_0, c_1, ..., c_{n-1}) \in C$.

Let $R_n = R_{u^4,v^2,2}[x]/ < x^n - 1 >$ and

$$\begin{array}{rcl} \pi & : & R^n_{u^4,v^2,2} \longrightarrow R_n \\ (c_0, c_1, ..., c_{n-1}) & \longmapsto & \pi((c_0, ..., c_{n-1})) = c_0 + c_1 x + c_{n-1} x^{n-1} (\mathrm{mod}(x^n - 1)) \end{array}$$

It is well known that C is a cyclic code of length n over $R_{u^4,v^2,2}$ if and only if $\pi(C)$ is an ideal of R_n.

3. The structures of cyclic codes over the ring $R_{u^4,v^2,2}$

In [4], the structure of a code of arbitrary length n over $R_{u^k,v^2,p}$, where k is a positive integer, p is a prime number was given. In according to that, by taking $k = 4, p = 2$, the code C over the ring $R_{u^4,v^2,2}$ is written as $\pi(C) =< A_1, ..., A_8 >$ where A_i' s are defined as follows;

$A_1 = g_1(x) + ug_{1,1}(x) + u^2 g_{1,2}(x) + u^3 g_{1,3}(x) + v(g_{1,4}(x) + ug_{1,5}(x) + u^2 g_{1,6}(x) + u^3 g_{1,7}(x))$

$A_2 = ug_2(x) + u^2 g_{2,2}(x) + u^3 g_{2,3}(x) + v(g_{2,4}(x) + ug_{2,5}(x) + u^2 g_{2,6}(x) + u^3 g_{2,7}(x))$

$A_3 = u^2 g_3(x) + u^3 g_{3,3}(x) + v(g_{3,4}(x) + ug_{3,5}(x) + u^2 g_{3,6}(x) + u^3 g_{3,7}(x))$

$A_4 = u^3 g_4(x) + v(g_{4,4}(x) + ug_{4,5}(x) + u^2 g_{4,6}(x) + u^3 g_{4,7}(x))$

$A_5 = v(g_5(x) + ug_{5,5}(x) + u^2 g_{5,6}(x) + u^3 g_{5,7}(x))$

$A_6 = v(ug_6(x) + u^2 g_{6,6}(x) + u^3 g_{6,7}(x))$

$A_7 = v(u^2 g_7(x) + u^3 g_{7,7}(x))$

$A_8 = vu^3 g_8(x)$

By using the Theorem 3.4 in [4], the following Theorem is written,

Theorem 2. *Let C be a cyclic code over the ring $R_{u^4,v^2,2}$ of length n. That is $\pi(C) =< A_1, ..., A_8 >$ is an ideal of R_n. If n is odd, then $\pi(C) =< g_1(x) + ug_2(x) + u^2 g_3(x) + u^3 g_4(x), v(g_5(x) + ug_6(x) + u^2 g_7(x) + u^3 g_8(x)) >$ where $g_4(x)|g_3(x)|g_2(x)|g_1(x)$, $g_8(x)|g_7(x)|g_6(x)|g_5(x)$, and $g_{k+i}(x)|g_i(x)$, for $i = 1, 2, 3, 4$.*

Theorem 3. *Let C be a cyclic code of odd length n over $R_{u^4,v^2,2}$. That is, $\pi(C) =< g_1(x) + ug_2(x) + u^2 g_3(x) + u^3 g_4(x), v(g_5(x) + ug_6(x) + u^2 g_7(x) + u^3 g_8(x)) >$ is ideal of R_n, where $g_4(x)|g_3(x)|g_2(x)|g_1(x)$, $g_8(x)|g_7(x)|g_6(x)|g_5(x)$, and $g_{k+i}(x)|g_i(x)$, for $i = 1, 2, 3, 4$. Then*

$$\pi(C) =< g_1(x), ug_2(x), u^2 g_3(x), u^3 g_4(x), vg_5(x), uvg_6(x), u^2 vg_7(x), u^3 vf_8(x) > .$$

Proof. Since $\pi(C) =< g_1(x)+ug_2(x)+u^2g_3(x)+u^3g_4(x), v(g_5(x)+ug_6(x)+u^2g_7(x)+u^3g_8(x)) >$, we get $\pi(C)(modv) =< g_1(x) + ug_2(x) + u^2g_3(x) + u^3g_4(x) >$. From Lemma 3.5 in [2], it is known that if n is odd, then $\pi(C)(modv) =< g_1(x)+ug_2(x)+u^2g_3(x) + u^3g_4(x) >=< g_1(x), ug_2(x), u^2g_3(x), u^3g_4(x) >$.

To show the other part, let us take $h(x) = vg_5(x)+uvg_6(x)+u^2vg_7(x)+u^3vg_8(x)$. We have $u^3h(x) = u^3vg_5(x)$ and $u^2(x^n - 1/g_5(x))h(x) = (x^n - 1/g_5(x))u^3vg_6(x) \in \pi(C)$. As n is odd, so $\left(\frac{x^n-1}{g_5(x)}, g_5(x)\right) = 1$. Hence there exist $p_1'(x), p_2'(x) \in Z_2[x]$ such that $\frac{x^n-1}{g_5(x)}p_1'(x) + g_5(x)p_2'(x) = 1$. By multiplying $u^3vg_6(x)$, we get $u^3vg_6(x) = u^3vg_6(x)\frac{x^n-1}{g_5(x)}p_1'(x)+u^3vg_6(x)g_5(x)p_2'(x)$. Since $u^3vg_5(x) \in \pi(C)$ and $u^3vg_6(x)\frac{x^n-1}{g_5(x)} \in \pi(C)$, we have $u^3vg_6(x) \in \pi(C)$. As n is odd, so $\left(\frac{x^n-1}{g_6(x)}, g_6(x)\right) = 1$. Hence there exist $s_1'(x), s_2'(x) \in F_2[x]$ such that $\frac{x^n-1}{g_6(x)}s_1'(x)+g_6(x)s_2'(x) = 1$. By multiplying $u^3vg_7(x)$, we get $u^3vg_7(x) = u^3vg_7(x)\frac{x^n-1}{g_6(x)}s_1'+u^3vg_7(x)g_6(x)s_2'(x)$. Since $u^3vg_6(x) \in \pi(C)$ and $u(x^n-1/g_6(x))h(x) = u^3vg_7(x)\frac{x^n-1}{g_6(x)} \in \pi(C)$, we have $u^3vg_7(x) \in \pi(C)$. As n is odd, so $\left(\frac{x^n-1}{g_7(x)}, g_7(x)\right) = 1$. Hence there exist $t_1'(x), t_2'(x) \in F_2[x]$ such that $\frac{x^n-1}{g_7(x)}t_1'(x) + g_7(x)t_2'(x) = 1$. By multiplying $u^3vg_8(x)$, we get $u^3vg_8(x) = u^3vg_7(x)\frac{x^n-1}{g_5(x)}t_1'(x) + u^3vg_8(x)g_7(x)t_2'(x)$. Since $u^3vg_7(x) \in \pi(C)$ and $(x^n - 1/g_7(x)) = u^3vg_8(x)\frac{x^n-1}{g_7(x)} \in \pi(C)$, we have $u^3vg_8(x) \in \pi(C)$. Similarly, it can be shown that $u^2vg_7(x), uvg_6(x) \in \pi(C)$. Since $vg_5(x)+uvg_6(x)+u^2vg_7(x)+u^3vg_8(x) \in \pi(C)$ and $uvg_6(x) \in \pi(C)$,$u^2vg_7(x) \in \pi(C)$,$u^3vg_8(x) \in \pi(C)$, then $vg_5(x) \in \pi(C)$. Therefore $\pi(C) =< g_1(x), ug_2(x), u^2g_{3(x)}$, $u^3g_4(x), vg_5(x), uvg_6(x), u^2vg_7(x), u^3vf_8(x) >$.

The other part is seen easily.

4. The reversible codes

In this section, the necessary and sufficient conditions for a cyclic code C over $R_{u^4,v^2,2}$ to be reversible are given.

Definition 1. *For a polynomial $f(x) = f_0 + ... + f_mx^m \in R_{u^4,v^2,2}[x]$ of degree m, the reciprocal of $f(x)$ is defined to be the polynomial $f(x)^* = x^m f(x^{-1})$. We note that $degf(x)^* \leq degf(x)$ and if $f_0 \neq 0$, then $degf(x)^* =degf(x)$, $f(x)$ is called self reciprocal if $f(x)^* = f(x)$.*

Lemma 4. *([1]) Let $s(x), p(x)$ be any two polynomials in with degree deg $s(x) \leq$ deg $p(x)$. Then*
1. $[p(x)s(x)]^* = p(x)^*s(x)^*$
2. $[p(x) + s(x)]^* = p(x)^* + x^is(x)^*$, where $i = deg\ p(x) - deg\ s(x)$

.

Definition 2. *The reverse of a codeword* $\alpha = (\alpha_0, \alpha_1, ..., \alpha_{n-1}) \in C$*, denoted by* α^r*, is defined* $\alpha^r = (\alpha_{n-1}, \alpha_{n-2}, ..., \alpha_0)$.

Definition 3. *A linear code* C *of length* n *over* $R_{u^4,v^2,2}$ *is said to be reversible if* $\alpha^r \in C$ *for all* $\alpha \in C$.

Theorem 5. *([6]) Let* $\pi(C) =< f(x) >$ *be an ideal of* $F_2[x]/ < x^n - 1 >$*, the* $\pi(C)$ *reversible if and only if* $f(x)$ *is self reciprocal.*

Theorem 6. *Let* C *be a cyclic code of odd length* n *over* $R_{u^4,v^2,2}$*. That is,*

$$\pi(C) =< g_1(x), ug_2(x), u^2g_3(x), u^3g_4(x), vg_5(x), uvg_6(x), u^2vg_7(x), u^3vg_8(x) >$$

where $g_4(x)|g_3(x)|g_2(x)|g_1(x)$*,* $g_8(x)|g_7(x)|g_6(x)|g_5(x)$ *and* $g_{k+i}(x)|g_i(x)$*, for* $i = 1, 2, 3, 4$*. Then* $\pi(C)$ *is reversible if and only if* $g_i(x)$ *are self reciprocal polynomials over* F_2*, where* $i = 1, ..., 8$.

Proof. Let C be a reversible code over $R_{u^4,v^2,2}$. We have $\pi(C)(modu) =< g_1(x), vg_5(x) >$. By using the Theorem 4.6 in [5], we have $g_1(x), g_5(x)$ are self reciprocal polynomials. It is shown that the polynomials $g_2(x), g_3(x), g_4(x), g_6(x)$, $g_7(x), g_8(x)$ are self reciprocal as in proof of Theorem 18 in [3].

Conversely, let $g_1(x), g_2(x), g_3(x), g_4(x), g_5(x), g_6(x), g_7(x), g_8(x)$ be self reciprocal polynomials over F_2. Let $c = (c_0, ..., c_{n-1}) \in C$. Then there exist $s_1(x), ..., s_8(x) \in F_2[x]$ such that $\pi(c) = c(x) = s_1(x)g_1(x) + us_2(x)g_2(x) + ... + u^3vs_8(x)g_8(x) \in \pi(C)$. Then we have

$c(x)^* = [s_1(x)g_1(x)+...+u^3vs_8(x)g_8(x)]^* = [s_1(x)g_1(x)+us_2(x)g_2(x)+u^2s_3(x)g_3(x)+u^3s_4(x)g_4(x)]^* + vx^j[s_5(x)g_5(x) + us_6(x)g_6(x) + u^2s_7(x)g_7(x) + u^3vs_8(x)g_8(x)]^* = [s_1(x)g_1(x) + us_2(x)g_2(x)]^* + u^2x^t[s_3(x)g_3(x) + us_4(x)g_4(x)]^* + vx^j\{[s_5(x)g_5(x) + us_6(x)g_6(x)]^* + u^2x^m[s_7(x)g_7(x) + us_8(x)g_8(x)]^*\} = s_1(x)^*g_1(x)^* + ux^as_2(x)^*g_2(x)^* + u^2x^t[s_3(x)^*g_3(x)^* + ux^bs_4(x)^*g_4(x)^*] + vx^j\{[s_5(x)^*g_5(x)^* + ux^ds_6(x)^*g_6(x)^*] + u^2x^m[s_7(x)^*g_7(x)^* + ux^ds_8(x)^*g_8(x)^*]\}$. By using the fact that $g_1(x)^* = g_1(x), g_2(x)^* = g_2(x), ..., g_8(x)^* = g_8(x)$, we get $c(x)^* = s_1(x)^*g_1(x) + ux^as_2(x)^*g_2(x) + u^2x^t[s_3(x)^*g_3(x) + ux^bs_4(x)^*g_4(x)] + vx^j\{[s_5(x)^*\ g_5(x) + ux^ds_6(x)^*g_6(x)] + u^2x^m[s_7(x)^*g_7(x) + ux^ds_8(x)^*g_8(x)]\} \in \pi(C)$, where j, t, m, a, b, d are as in Lemma 4. Hence $\pi(C)$ is reversible code.

5. A Correspondence Between the Elements of the Ring $R_{u^4,v^2,2}$ and DNA 4-bases

Let $S_{D_4} = \{A, T, C, G\}$ *represent the DNA alphabet. We use the same notation for the set* $S_{D_{16}} = \{AA, TA, ..., GG\}$ *and* $S_{D_{256}} = \{AAAA, TTTT, ..., GGGG\}$.

The Watson Crick Complement is given $\overline{A} = T, \overline{T} = A, \overline{C} = G, \overline{G} = C$. *Naturally we extend this notion to the elements of* $S_{D_{16}}$ *and* $S_{D_{256}}$ *such that* $\overline{AA} = TT, ..., \overline{TG} = AC$ *and* $\overline{AAAA} = TTTT, ..., \overline{GGGG} = CCCC$, *respectively.*

We define one to one correspondence ξ_1 *between* $R_{u^4,2}$ *and* $S_{D_{16}}$ *as follows.*

Elements α	*DNA double pairs* $\xi_1(\alpha)$
0	AA
1	AG
$u+1$	AT
u	AC
u^2+1	GG
$1+u+u^2+u^3$	TT
u^2	GA
u^3+u^2+u	TC
$u+u^2$	GC
u^2+u^3	TA
$u+u^3$	CC
u^3	CA
$1+u+u^3$	CT
$1+u+u^2$	GT
$1+u^2+u^3$	TG
$1+u^3$	CG

The map ξ_1 *can be extended to* $(R_{u^4,2})^n$, *naturally.*

For instance $(c_0, c_1, c_2) = (1, u, 1+u) \in R^3_{u^4,2}$ *is mapped to* $\xi_1((1, u, 1+u)) = (AG, AC, AT)$.

By using ϕ and the following map

$$\begin{array}{rcl} \gamma & : & (R_{u^4,2})^2 \longrightarrow (S_{D_4})^4 = S_{D_{256}} \\ (e,f) & \longmapsto & \gamma(e,f) = (\xi_1(e), \xi_1(f)) \end{array}$$

we get one to one correspondence ξ_2 between $R_{u^4,v^2,2}$ and $S_{D_{256}}$ as follows.

$$\xi_2 \ : \ R_{u^4,v^2,2} \longrightarrow (R_{u^4,2})^2 \longrightarrow (S_{D_4})^4 = S_{D_{256}}$$

$$\beta = x + vy \longmapsto (x, x+y) \longmapsto (\xi_1(x), \xi_1(x+y)) \leftrightarrow \xi_1(x)\xi_1(x+y)$$

where $x, y \in R_{u^4,2}$ and $\xi_2 = \gamma\phi$.

Elements β	Gray images	DNA 4-bases $\xi_2(\beta)$
0	$(0,0)$	$AAAA$
1	$(1,1)$	$AGAG$
$(1+u)+v(1+u+u^2)$	$(1+u,u^2)$	$ATGA$
$\vdots$	$\vdots$	$\vdots$

The map ξ_2 can be extended to $(R_{u^4,v^2,2})^n$, naturally.

For instance $(c_0,c_1,c_2)=\big(1+u,vu,1+u+u^2\big)\in R^3_{u^4,v^2,2}$ is mapped to

$$\xi_2(\big(1+u,vu,1+u+u^2\big))=(ATAT,AAAC,GTGT).$$

6. The reversible complement codes

In this section, the necessary and sufficient conditions for a cyclic code C over $R_{u^4,v^2,2}$ to be reversible complement are given.

Lemma 7. *For any $a\in R_{u^4,v^2,2}$, we have $a+\overline{a}=1+u+u^2+u^3$, where $\overline{a}$ represents the complement of a in $R_{u^4,v^2,2}$.*

Definition 4. *The reversible complement of a codewords $d=(d_0,d_1,...,d_{n-1})\in C$, denoted by d^{rc}, is defined $d^{rc}=\big(\overline{d}_{n-1},\overline{d}_{n-2},...,\overline{d}_1,\overline{d}_0\big)$, where $\overline{d}_i$ is the complement of d_i for $i=0,1,..,n-1$.*

Definition 5. *A linear code C of length n over $R_{u^4,v^2,2}$ is said to be reversible complement if $d^{rc}\in C$ for every $d\in C$.*

Theorem 8. *Let C be a cyclic code of odd length n over $R_{u^4,v^2,2}$. That is,*

$$\pi(C)=<g_1(x),ug_2(x),u^2g_3(x),u^3g_4(x),vg_5(x),uvg_6(x),u^2vg_7(x),u^3vg_8(x)>$$

where $g_4(x)|g_3(x)|g_2(x)|g_1(x)$, $g_8(x)|g_7(x)|g_6(x)|g_5(x)$ and $g_{k+i}(x)|g_i(x)$, for $i=1,2,3,4$. Then $\pi(C)$ is reversible complement if and only if $g_i(x)$ are self reciprocal polynomials over F_2, where $i=1,...,8$ and $(1+u+u^2+u^3)(1+x+...+x^{n-1})\in\pi(C)$.

Proof. Let $\pi(C)$ be reversible complement code. Then since $\mathbf{0}=0+0x+...+0x^{n-1}\in\pi(C)$, then $\mathbf{0}^{rc}=(1+u+u^2+u^3)(1+x+...+x^{n-1})\in\pi(C)$. In order to show that $g_1(x)^*=g_1(x)$, take $g_1(x)=1+a_1x+...+a_{r-1}x^{r-1}+x^r$. So $g_1(x)^{rc}=\overline{0}+\overline{0}x+...+\overline{0}x^{n-r-2}+\overline{1}x^{n-r-1}+\overline{a_{r-1}}x^{n-r}+...+\overline{a_1}x^{n-2}+\overline{1}x^{n-1}\in\pi(C)$. Since $(1+u+u^2+u^3)(x^n-1/x-1)\in\pi(C)$, the poynomial $g_1(x)^{rc}+(1+u+u^2+u^3)(x^n-1/x-1)$ is equal the poynomial $x^{n-r-1}g_1(x)^*\in\pi(C)$.

So $g_1(x)^* \in \pi(C)$. From this, there exist $s_1(x), ..., s_8(x) \in R_{u^4,v^2,2}[x]$ such that $g_1(x)^* = g_1(x)s_1(x) + ug_2(x)s_2(x) + ... + u^3vg_8(x)s_8(x)$. As $g_1(x) \in F_2[x], g_1(x)$ is a monic, so we get $s_1(x) = 1, s_i(x) = 0$ for $i = 2, .., 8$. Therefore $g_1(x)^* = g_1(x)$. Similarly it is shown that $g_2(x), .., g_8(x)$ to be self reciprocal.

Conversely, let $g_i(x)$ are self reciprocal polynomials over F_2, where $i = 1, ..., 8$ and $(1 + u + u^2 + u^3)(1 + x + ... + x^{n-1}) \in \pi(C)$. Take $d(x) = d_0 + d_1x + ... + d_kx^k \in \pi(C)$. So there exist $a_1(x), ..., a_8(x) \in R_{u^4,v^2,2}[x]$ such that $d(x) = g_1(x)a_1(x)+ug_2(x)a_2(x)+...+u^3vg_8(x)a_8(x)$. From it, we get $d(x)^* = g_1(x)^*a_1(x)^*+ ux^{t_1}g_2(x)^*a_2(x)^* + ... + u^3vx^{t_7}g_8(x)^*a_8(x)^*$ where $t_1, .., t_7$ are as in Lemma 4. By using $g_1(x)^* = g_1(x), ..,$g$_8(x)^* = g_8(x)$, we get $d(x)^* \in \pi(C)$. Since $\pi(C)$ linear, we get $(1 + u + u^2 + u^3)(x^n - 1/x - 1) + x^{n-k-1}d(x) = \{d(x)^*\}^{rc} \in \pi(C)$. It is known that $\{\{d(x)^*\}^{rc}\}^* = d(x)^{rc} \in \pi(C)$. So $\pi(C)$ is reversible complement.

By a cyclic DNA code over $R_{u^4,v^2,2}$ of length n, we mean a cyclic code C that has the reverse complement property. i.e. C is a cyclic DNA if C is a cyclic and $c = (c_0, c_1, ..., c_{n-1}) \in C$ implies $c^{rc} = (\overline{c_{n-1}}, \overline{c_{n-2}}, ..., \overline{c_0}) \in C$, where $\overline{c_i}$ stands for the complement of c_i in $R_{u^4,v^2,2}$ for $i = 0, .., n-1$. Equivalently, a cyclic DNA code C means that $\pi(C)$ is an ideal of R_n and $f(x) = c_0+c_1x+...+c_{n-1}x^{n-1} \in \pi(C)$ implies $f(x)^{rc} = \overline{c_{n-1}} + \overline{c_{n-2}}x + ... + \overline{c_0}x^{n-1} \in \pi(C)$, where $\overline{c_i}$ stands for the complement of c_i in $R_{u^4,v^2,2}$ for $i = 0, ..., n - 1$.

By using the correspondence ξ_2 and cyclic DNA codes, we obtain DNA codes as follows.

Theorem 9. *Let C cyclic DNA code of length n over $R_{u^4,v^2,2}$ with minimum distance d. That is, $\pi(C) =< g_1(x), ug_2(x), u^2g_3(x), u^3g_4(x), vg_5(x), uvg_6(x), u^2vf_7(x), u^3vf_8(x) >$ is an ideal of R_n and $\pi(C)$ has reversible complement property. Then $\xi_2(C)$ is a DNA code of length n over the alphabet $\{AAAA, TTTT, ..., GGGG\}$ with the minimum Hamming distance at least d.*

7. Conclusion

In this work, the necessary and sufficient conditions for cyclic codes of odd length over the ring $R_{u^4,v^2,2}$ to be reversible complement are given and DNA codes are obtained via cyclic DNA codes of odd length n over the finite ring $R_{u^4,v^2,2}$.

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