

SUBORDINATION AND RELATED RESULTS ON FUNCTIONS ASSOCIATED WITH (J, K) -SYMMETRICAL POINTS

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ABSTRACT. In this study, we make use of some first order differential subordination and superordination conditions on functions associated with (j, k) -symmetrical points, and determine conditions for some already known classes of analytic functions. We're also come up with some sandwich theorems based on specific assumptions about the parameters that are involved in our major findings.

2010 *Mathematics Subject Classification:* 30C45, 30C80.

Keywords: Functions with (j, k) -symmetrical points, subordination.

1. INTRODUCTION

Let $\mathbb{E} : \mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$ contained in \mathbb{C} and let $\mathcal{H}(\mathbb{E})$ be the family of all mappings holomorphic in \mathbb{E} . For holomorphic mappings $f, g \in \mathcal{H}(\mathbb{E})$, $f \prec g$, if for a Schwarz mapping w , we write that $f(z) = g(w(z))$, for every $z \in \mathbb{E}$. If $g \in \mathcal{S}$, then $f \prec g \iff g(0) = f(0)$ and $g(\mathbb{E}) \supset f(\mathbb{E})$. Assume that \mathcal{P} contains the family of holomorphic or analytic mappings p , in such a way that $p \in \mathcal{H}(\mathbb{E}) : p(0) = 1, \Re(p(z)) > 0$ and

$$p(z) = 1 + c_1 z + \dots, z \in \mathbb{E}.$$

For $-1 \leq \alpha_2 < \alpha_1 \leq 1$, we define that a mapping $p \in \mathcal{P}[\alpha_1, \alpha_2] : p(0) = 1$ and $p(z) \prec \frac{1+\alpha_1 z}{1+\alpha_2 z}$. We refer to [9] for more information. Consider that \mathcal{A} denote the family or class of holomorphic or analytic mappings defined in \mathbb{E} and meet the assumption that $f(0) = 0$ and $f'(0) = 1$. A mapping $f \in \mathcal{A}$ considers the following illustration:

$$f(z) = z + a_2 z^2 + \dots, z \in \mathbb{E}. \quad (1)$$

A mapping $f \in \mathcal{A}$ is considered to be univalent, if it is one-to-one in \mathbb{E} . This family is abbreviated by \mathcal{S} . A mapping $f \in \mathcal{S}^*$ in \mathbb{E} is said to be starlike assuming that

$$\Re \left(\frac{z f'(z)}{f(z)} \right) > 0.$$

This condition is due to Nevanlinna. A mapping f is convex, iff $zf' \in \mathcal{S}^*$. This condition was studied by Study. Lowner and many others also studied these families at breadth. In addition, Ma and Minda also defined the following:

$$f \in \mathcal{S}^*(\varphi) \iff \frac{zf'(z)}{f(z)} \prec \varphi(z), \text{ where } \varphi \in \mathcal{P}, z \in \mathbb{E}.$$

For detail, see [8, 17] with reference therein.

Definition 1. A mapping $f \in \mathcal{S}_{\text{SP}}$ in \mathbb{E} iff it fulfills the condition:

$$\Re \left(\frac{zf'(z)}{f(z) - f(-z)} \right) > 0.$$

Stankiewicz defined the related family \mathcal{C}_{SP} of convex mappings.

Definition 2. A mapping $f \in \mathcal{S}_{\text{SCP}}$ iff it observes the condition:

$$\Re \left(\frac{zf'(z)}{f(z) - \overline{f(-\bar{z})}} \right) > 0.$$

For detail see [7]. Also, $f \in \mathcal{C}_{\text{SCP}}$ iff $zf' \in \mathcal{S}_{\text{SCP}}$. For more detail, see [10].

Definition 3. A mapping $f \in \mathcal{S}_{\text{SCP}}(\varphi)$, if we have

$$\frac{zf'(z)}{f(z) - \overline{f(-\bar{z})}} \prec \varphi(z), \text{ where } z \in \mathbb{E} \text{ and } \varphi \in \mathcal{P}.$$

Also a mapping $f \in \mathcal{C}_{\text{SCP}}(\varphi)$, iff, $zf' \in \mathcal{S}_{\text{SCP}}(\varphi)$. These families were first studied by Ravichandran [16] in 2004. The families $\mathcal{S}_{\text{SP}}^k(\varphi)$ and $\mathcal{C}_{\text{SP}}^k(\varphi)$, investigated by Wang and Gao [21] are defined as:

$$\mathcal{S}_{\text{SCP}}^k(\varphi) = \left\{ f \in \mathcal{S}, \frac{zf'(z)}{f_k(z)} \prec \varphi(z), \text{ where } \varphi \in \mathcal{P}, k \neq 1, k \in \mathbb{Z}^+, z \in \mathbb{E} \right\}$$

and $f \in \mathcal{C}_{\text{SCP}}^k(\psi)$ iff $zf' \in \mathcal{S}_{\text{SCP}}^k(\varphi)$.

$$\mathcal{C}_{\text{SCP}}^k(\varphi) = \left\{ f \in \mathcal{S}, \frac{(zf'(z))'}{f_k'(z)} \prec \varphi(z), \text{ where } \varphi \in \mathcal{P}, k \neq 1, k \in \mathbb{Z}^+, z \in \mathbb{E} \right\},$$

A mapping $f \in \mathcal{S}_{\text{SCP}}^{2k}$, iff it meets the condition:

$$\Re \frac{zf'(z)}{f_{2k}(z)} > 0, z \in \mathbb{E}.$$

where $k \neq 1$ and $k \in \mathbb{Z}^+$. Also, $f \in \mathcal{S}_{\text{SCP}}^{2k}(\phi)$, iff it meets the condition:

$$\frac{zf'(z)}{f_{2k}(z)} \prec \phi(z)$$

where $\phi \in \mathcal{P}$. Also, $f \in \mathcal{C}_{\text{SCP}}^{2k}(\phi)$, iff: $zf' \in \mathcal{S}_{\text{SCP}}^{2k}(\phi)$. The families $\mathcal{S}_{\text{SCP}}^{2k}(\phi)$ and $\mathcal{C}_{\text{SCP}}^{2k}(\phi)$ were investigated by Wang and Geo. See [21].

Let $k \in \mathbb{N}$, $j = 0, 1, \dots, k-1$. A mapping f is a (j, k) -symmetrical if for each $z \in \mathbb{E}$, $f(\epsilon z) = \epsilon^j f(z)$, where $\epsilon = \exp \frac{2\pi i}{k}$. For detail, see [11]. Observe that $f_{j,k}$ is defined by:

$$f_{j,k}(z) = \frac{1}{k} \sum_{\eta_1=0}^{k-1} \frac{f(\epsilon^{\eta_1} z)}{\epsilon^{\eta_1 j}}. \quad (2)$$

This mappings were first studied by Liczberski and Polubinski in [12]. From (2), we see that

$$f_{j,k}(\epsilon^{\eta_1 j} z) = \epsilon^{\eta_1 j} f_{j,k}(z),$$

$$f'_{j,k}(\epsilon^{\eta_1 j} z) = \epsilon^{\eta_1 j - \eta_1} f'_{j,k}(z) = \frac{1}{k} \sum_{\eta_1=0}^{k-1} \frac{f'(\epsilon^{\eta_1} z)}{\epsilon^{\eta_1 j - 2\eta_1}},$$

and

$$f''_{j,k}(\epsilon^{\eta_1 j} z) = \epsilon^{\eta_1 j - 2\eta_1} f''_{j,k}(z) = \frac{1}{k} \sum_{\eta_1=0}^{k-1} \frac{f''(\epsilon^{\eta_1} z)}{\epsilon^{\eta_1 j - 2\eta_1}}.$$

Definition 4. A mapping $f \in \mathcal{S}_{\text{SCP}}^{j,k}$ in \mathbb{E} , iff it meets the condition:

$$\Re \left(\frac{zf'(z)}{f_{j,k}(z)} \right) > 0.$$

Similarly, a mapping $f \in \mathcal{C}_{\text{SCP}}^{j,k}$, iff, $zf' \in \mathcal{S}_{\text{SCP}}^{j,k}$. Also, we define the family $\mathcal{S}_{\text{SCP}}^{j,k}(\phi)$ as follows:

Definition 5. A mapping $f \in \mathcal{S}_{\text{SCP}}^{j,k}(\phi)$ in \mathbb{E} , iff it meets the condition:

$$\Re \left(\frac{zf'(z)}{f_{j,k}(z)} \right) \prec \phi(z).$$

Similarly, $f \in \mathcal{C}_{\text{SCP}}^{j,k}(\phi)$ iff, $zf' \in \mathcal{S}_{\text{SCP}}^{j,k}(\phi)$, see [12]. Let $\Psi : \mathbb{C}^3 \times \mathbb{E} \rightarrow \mathbb{C}$ and $p \in \mathcal{S}$ in \mathbb{E} . If h holomorphic in \mathbb{E} in such a way that

$$\Psi(h, zh', h''; z) \prec p(z), \quad (3)$$

then h is the solution of (3) and $\ell \in \mathcal{S}$ dominated h if $h(z) \prec \ell(z)$ for h satisfying (3). Then $\tilde{\ell}$ best dominated (3), if $\tilde{\ell}(z) \prec \ell(z)$, for every ℓ satisfying (3). For detail, see [13]. On the other side, we also have

$$p(z) \prec \Psi(h, zh', h''; z) \quad (4)$$

then h is again a solution. A mapping ℓ is called subordinate of h , if $\ell(z) \prec h(z)$ for h satisfying (4). A mapping $\tilde{\ell}$ in such a way that $\ell(z) \prec \tilde{\ell}(z)$ for every subordinate ℓ satisfying (4) is best subordinate of (4). For more detailed information, see [14].

Using similar results of Bulboaca [6], Aouf et al. [1] proved that

$$\ell_1(z) \prec \frac{zf'(z)}{f(z)} \prec \ell_2(z),$$

where $\ell_1 : \ell_1(0) = 1, \ell_2 : \ell_2(0) = 1$ are in \mathcal{S} . In 2006, Shanmugam et.al [19, 20] further extended these results, see [2, 3, 4, 5, 15, 18] for further information on these and other relevant families.

The main goal of this research is to look at certain analogues of differential inequalities for mappings to j, k -symmetric points and come up with some sandwich results for these mappings.

2. A SET OF PRELIMINARY RESULTS

Lemma 1. [13] For $\ell \in \mathcal{S}$ in the domain \mathbb{E} , and θ and ψ are holomorphic or analytic in $\mathbb{D} : \ell(\mathbb{E}) \subset \mathbb{D}$ and for $w \in \ell(\mathbb{E})$, $\psi(w) \neq 0$. We define and establish that

$$Q(z) = z\psi[\ell(z)]\ell'(z),$$

along with

$$h(z) = \theta[\ell(z)] + Q(z)$$

and further note that $h \in \mathcal{C}$ or $Q \in \mathcal{S}^*$ in \mathbb{E} in such a way that $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$. If p is holomorphic in $\mathbb{E} : p(0) = \ell(0), p(\mathbb{E}) \subset \mathbb{D}$ and we have the subordination given by

$$zp'(z)\psi[p(z)] + \theta[p(z)] \prec z\ell'(z)\psi[\ell(z)] + \theta[\ell(z)] = h(z), \quad (5)$$

then $p(z) \prec \ell(z)$.

Let \mathcal{Q} be the family of holomorphic or analytic and injective mappings f on $\overline{\mathbb{E}} \setminus \partial\mathbb{E}(f)$, where

$$\partial\mathbb{E}(f) = \left\{ \zeta \in \partial\mathbb{E} \text{ such that } \lim_{z \rightarrow \zeta} f(z) = \infty, f'(\zeta) \neq 0 \text{ and } \zeta \in \partial\mathbb{E} \setminus \partial\mathbb{E}(f) \right\}.$$

Lemma 2. [13] For $\ell \in \mathcal{S}$ in the domain \mathbb{E} , and θ and ψ are holomorphic or analytic in a domain $\mathbb{D} : \ell(\mathbb{E}) \subset \mathbb{D}$ with $\psi(w) \neq 0, w \in \ell(\mathbb{E})$, assume that

$$\Re \left(\frac{\theta'[\ell(z)]}{\psi[\ell(z)]} \right) > 0, z \in \mathbb{E}$$

and

$$h(z) = z\ell'(z)\psi[\ell(z)] \in \mathcal{S}^*.$$

If $p \in \mathcal{H}[\ell(0), 1] \cap \mathcal{Q}$ with $p(\mathbb{E}) \subset \mathbb{D}$,

$$zp'(z)\psi[p(z)] + \theta[p(z)] \in \mathcal{S}$$

and

$$z\ell'(z)\psi[\ell(z)] + \theta[\ell(z)] \prec \theta(p(z)) + zp'(z)\psi[p(z)], \quad (6)$$

then $\ell(z) \prec (p(z))$, and ℓ is the best subordinated by (6).

Lemma 3. [8] The mapping

$$\ell(z) = \frac{1}{(1-z)^{2ab}} \in \mathcal{S} \iff |2ab \pm 1| \leq 1.$$

3. SUBORDINATION RESULTS

Theorem 4. Assume that a mapping $\ell \in \mathcal{S} : \ell(0) = 1$ defined in \mathbb{E} and fulfills the inequality

$$\Re \left(1 + \frac{z\ell''(z)}{\ell'(z)} \right) > \max \left\{ 0; -\Re \left(\frac{\eta}{\gamma} \right) \right\}, \gamma \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}, z \in \mathbb{E}. \quad (7)$$

If for $f \in \mathcal{A}$, the non-linear differential equation given by

$$(\eta + \gamma) \frac{zf'(z)}{f_{j,k}(z)} + \gamma \frac{z^2 f''(z)}{f_{j,k}(z)} - \gamma \frac{z^2 f'(z) f'_{j,k}(z)}{(f_{j,k}(z))^2} = \gamma z [\ell(\omega(z))]' + \eta \ell(\omega(z)), z \in \mathbb{E}, \quad (8)$$

holds, then $\frac{zf'(z)}{f_{j,k}(z)} \prec \ell(z), z \in \mathbb{E}$.

Proof. In order to acquire the appropriate proof, we define and examine the following functional h in such a way that

$$h(z) = \frac{zf'(z)}{f_{j,k}(z)}, \quad (9)$$

where $h : h(0) = 1$ is holomorphic in \mathbb{E} . On taking derivative of (8), we can write

$$h'(z) = \frac{-zf'(z)f_{j,k}(z) + f_{j,k}(z)\{f'(z) + zf''(z)\}}{(f_{j,k}(z))^2}, \quad z \in \mathbb{E},$$

which further implies that

$$\eta h(z) + \gamma zh'(z) = \gamma \frac{z^2 f''(z)}{f_{j,k}(z)} + (\eta + \gamma) \frac{zf'(z)}{f_{j,k}(z)} - \gamma \frac{z^2 f'(z) f'_{j,k}(z)}{(f_{j,k}(z))^2}. \quad (10)$$

In view of (8), from (10) it is observed that

$$\eta h(z) + \gamma zh'(z) \prec \eta \ell(z) + \gamma z \ell'(z).$$

For the sack of the completeness of the condition of the Lemma 1, we set $\theta(w) = \eta w$ and $\psi(w) = \gamma$ which are holomorphic or analytic in \mathbb{C}^* in such a way that $\ell(\mathbb{E}) \subset \mathbb{C}^*$. Also, if we assume that

$$Q(z) = \psi[\ell(z)] z \ell'(z) = \gamma z \ell'(z)$$

along with

$$p(z) = Q(z) + \theta[\ell(z)] = \eta \ell(z) + \gamma z \ell'(z),$$

so that $Q(0) = 0$ and $Q'(0) = \gamma \ell'(0) \neq 0$, then from (7), it is clear that the mapping $Q \in \mathcal{S}^*$ in \mathbb{E} and

$$\Re \left(\frac{zp'(z)}{Q(z)} \right) = \Re \left(1 + \frac{\eta}{\gamma} + \frac{z\ell''(z)}{\ell'(z)} \right) > 0 \quad (z \in \mathbb{E}).$$

Making use of (8) along with Lemma 1, we find that $h(z) \prec \ell(z)$. This leads to the proof.

Remark 1. If we consider $\ell(z) = \frac{1+\alpha_1 z}{1+\alpha_2 z}$ in Theorem 4, where $-1 \leq \alpha_2 < \alpha_1 \leq 1$, then from the condition (7), we can write

$$\Re \left(\frac{1 - \alpha_2 z}{1 + \alpha_2 z} \right) > \max \left\{ 0; -\Re \left(\frac{\eta}{\gamma} \right) \right\} \quad (z \in \mathbb{E}).$$

Also the mapping $w(z) = \frac{1-z}{1+z} \in \mathcal{C}$ for $|z| < |\alpha_2|$ in \mathbb{E} and also $w(\bar{z}) = \overline{w(z)}$ for all $|z| < |\alpha_2|$. It implies that $w(\mathbb{E})$ is convex as well as symmetric about the real axis. Hence

$$\inf \left\{ \Re \left(\frac{1 - \alpha_2 z}{1 + \alpha_2 z} \right), z \in \mathbb{E} \right\} = \frac{1 - |\alpha_2|}{1 + |\alpha_2|} > 0.$$

Thus, we have $\Re \left(\frac{\eta}{\gamma} \right) \geq \frac{|\alpha_2| - 1}{|\alpha_2| + 1}$.

In view of the Theorem 4, we may derive:

Corollary 5. Suppose that $\eta, \gamma \in \mathbb{C} : \gamma \neq 0$ and $-1 \leq \alpha_2 < \alpha_1 \leq 1$ so that $\frac{1 - |\alpha_2|}{1 + |\alpha_2|} \geq \max \left\{ 0; -\Re \left(\frac{\eta}{\gamma} \right) \right\}$. If for the mapping $f \in \mathcal{A}$, we note the following condition

$$\gamma \frac{z^2 f''(z)}{f_{j,k}(z)} + (\eta + \gamma) \frac{z f'(z)}{f_{j,k}(z)} - \gamma \frac{z^2 f'(z) f'_{j,k}(z)}{(f_{j,k}(z))^2} \prec \eta \frac{1 + \alpha_1 z}{1 + \alpha_2 z} + \gamma \frac{(\alpha_1 - \alpha_2) z}{(1 + \alpha_2 z)^2}, z \in \mathbb{E},$$

then $\frac{z f'(z)}{f_{j,k}(z)} \in \mathcal{P}[\alpha_1, \alpha_2], z \in \mathbb{E}$.

For $\alpha_1 = 1$ and $\alpha_2 = -1$, we may derive the related results as a special case.

Theorem 6. Suppose that a mapping $\ell \in \mathcal{S}$ defined in \mathbb{E} in such a way that $\ell(0) = 1$ and we further assume that $\ell(z) \neq 0$. Consider that the condition

$$\Re \left(1 + \frac{z \ell''(z)}{\ell'(z)} - \frac{z \ell'(z)}{\ell(z)} \right) > 0 \quad (z \in \mathbb{E}). \quad (11)$$

holds for ℓ . Assume that for the mapping f given by (1) and $\eta \in \mathbb{C}^*$ and $\eta_1, \eta_2 \in \mathbb{C}$ with $\eta_1 + \eta_2 \neq 0$, we have

$$\frac{(\eta_1 + \eta_2) z f'(z)}{\eta_1 z f'_{j,k}(z) + \eta_2 f_{j,k}(z)} \neq 0 \quad (z \in \mathbb{E}). \quad (12)$$

If we see the subordination given by

$$2 + \frac{z f''(z)}{f'(z)} - \frac{\eta_1 z (z f'_{j,k}(z))' + \eta_2 z f'_{j,k}(z)}{\eta_1 z f'_{j,k}(z) + \eta_2 f_{j,k}(z)} \prec 1 + \frac{z \ell'(z)}{\eta \ell(z)} \quad (z \in \mathbb{E}), \quad (13)$$

holds, then we get $\left[\frac{(\eta_1 + \eta_2) z f'(z)}{\eta_1 z f'_{j,k}(z) + \eta_2 f_{j,k}(z)} \right]^\eta \prec \ell(z)$.

Proof. In order to acquire the appropriate proof, we examine the functional h in such a way that

$$h(z) = \left[\frac{(\eta_1 + \eta_2) z f'(z)}{\eta_1 z f'_{j,k}(z) + \eta_2 f_{j,k}(z)} \right]^\eta \quad (z \in \mathbb{E}), \quad (14)$$

where the power is assumed as the principal one. According to the supposition (12), the multivalued power mapping h has an holomorphic or analytic branch in \mathbb{E} , with $h(0) = 1$, and from (14), we can write that

$$2 + \frac{zf''(z)}{f'(z)} - \frac{\eta_1 z(zf'_{j,k}(z))' + \eta_2 z f'_{j,k}(z)}{\eta_1 z f'_{j,k}(z) + \eta_2 f_{j,k}(z)} = \frac{zh'(z)}{\eta h(z)} + 1 \quad (z \in \mathbb{E}). \quad (15)$$

In view of (13) and (15), we observe that

$$1 + \frac{zh'(z)}{\eta h(z)} \prec 1 + \frac{z\ell'(z)}{\eta \ell(z)} \quad (z \in \mathbb{E}).$$

When it comes to Lemma 1, we assume that $\theta(w) = 1$ and $\psi(w) = 1/\eta w$ are holomorphic or analytic in \mathbb{E} . These holomorphic or analytic mappings in \mathbb{C}^* containing $\ell(\mathbb{E})$. Furthermore, if we assume that

$$Q(z) := z\ell'(z)\psi[\ell(z)] = \frac{z\ell'(z)}{\eta \ell(z)}$$

and

$$p(z) := \theta[\ell(z)] + Q(z) = 1 + \frac{z\ell'(z)}{\eta \ell(z)}$$

with $Q(0) = 0$ and $Q'(0) = \frac{1}{\eta} \frac{\ell'(0)}{\ell(0)} \neq 0$, then from our assumption (11), it is proved that $Q \in \mathcal{S}^*$ in \mathbb{E} and

$$\Re\left(\frac{zp'(z)}{Q(z)}\right) = \Re\left(1 - \frac{z\ell'(z)}{\ell(z)} + \frac{z\ell''(z)}{\ell'(z)}\right) > 0 \quad (z \in \mathbb{E}).$$

Lemma 1, and the assumption given by (13), proves that $h(z) \prec \ell(z)$.

As a particular case, if we choose $\eta_1 = 0, \eta_2 = 1$ and $\ell(z) = \frac{1+\alpha_1 z}{1+\alpha_2 z}$ in the above theorem, then clearly the inequality (13) holds whenever $-1 \leq \alpha_1 < \alpha_2 \leq 1$. Hence, we can derive the following corollaries:

Corollary 7. *Let $-1 \leq \alpha_1 < \alpha_2 \leq 1$ and $\eta \in \mathbb{C}^*$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition*

$$\frac{zf'(z)}{f_{j,k}(z)} \neq 0 \quad (z \in \mathbb{E}). \quad (16)$$

If we write

$$2 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{j,k}(z)}{f_{j,k}(z)} \prec 1 + \frac{1}{\eta} \frac{(\alpha_1 - \alpha_2)z}{(1 + \alpha_1 z)(1 + \alpha_2 z)} \quad (z \in \mathbb{E}), \quad (17)$$

then we obtain $\left(\frac{zf'(z)}{f_{j,k}(z)}\right)^\eta \prec \frac{1+\alpha_1 z}{1+\alpha_2 z}$.

Corollary 8. Let $-1 \leq \alpha_1 < \alpha_2 \leq 1$ with $\alpha_2 \neq 0$ and suppose that $|\frac{\alpha_1 - \alpha_2}{\alpha_2 - 1}| \leq \frac{1}{\eta}$ or $|\frac{\alpha_1 - \alpha_2}{\alpha_2 + 1}| \leq \frac{1}{\eta}$ where $\eta \in \mathbb{C}^*$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition in (16) in such a way that the subordination given by

$$1 + \eta \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{j,k}(z)}{f_{j,k}(z)} \right] \prec \frac{1 + [\eta(\alpha_1 - \alpha_2) + \alpha_2]z}{1 + \alpha_2 z} \quad (z \in \mathbb{E}), \quad (18)$$

holds, then we observe that $\left(\frac{zf'(z)}{f_{j,k}(z)}\right)^\eta \prec (1 + \alpha_2 z)^{\frac{\eta(\alpha_1 - \alpha_2)}{\alpha_2}}$.

Using $\eta_1 = 0$, $\eta_2 = 1$ with $a, b \in \mathbb{C}^*$, $\eta = ab$ and $\ell(z) = \frac{1}{(1-z)^{2ab}}$ in the above Theorem 6 together with application of the Lemma 3, we arrive at the following conclusion as a corollary:

Corollary 9. Let $a, b \in \mathbb{C}^*$ in such a way that $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition in (16) in such a way that

$$2 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{j,k}(z)}{f_{j,k}(z)} \prec \frac{1+z}{1-z}$$

then we see that

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1}{(1-z)^2}. \quad (19)$$

By putting $\eta_1 = 0$, $\eta_2 = 1$, $\eta = \frac{e^{i\pi}}{b \cos \lambda}$, $a, b \in \mathbb{C}^*$, $|\lambda| < \frac{\pi}{2}$, and $\ell(z) = \frac{1}{(1-z)^{2abe^{-i\lambda} \cos \lambda}}$ in the above Theorem 6, we obtain the following:

Corollary 10. Assume that $a, b \in \mathbb{C}^*$ and $|\lambda| < \frac{\pi}{2}$, and suppose that

$$|2abe^{-i\lambda} \cos \lambda - 1| \leq 1 \quad \text{or} \quad |2abe^{-i\lambda} \cos \lambda + 1| \leq 1.$$

Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition in (16), and

$$1 + \frac{e^{i\pi}}{b \cos \lambda} \left[1 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{j,k}(z)}{f_{j,k}(z)} \right] \prec \frac{1+z}{1-z}$$

then,

$$\frac{zf'(z)}{f_{j,k}(z)} \prec \frac{1}{(1-z)^{2ab}}.$$

Theorem 11. *Suppose that the mapping $\ell \in \mathcal{S} : \ell(0) = 1$ and it meets the condition*

$$\Re \left(1 + \frac{z\ell''(z)}{\ell'(z)} \right) > \max \left\{ 0; -\Re \left(\frac{\delta}{\eta} \right) \right\}, \quad (20)$$

where $\eta, \gamma \in \mathbb{C}^*$ and $\eta_1, \eta_2 \in \mathbb{C}$ with $\eta_1 + \eta_2 \neq 0$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition in (1) and is in such a way that

$$\frac{(\eta_1 + \eta_2)zf'(z)}{\eta_1zf'_{j,k}(z) + \eta_2f_{j,k}(z)} \neq 0 \quad (z \in \mathbb{E}),$$

and if the subordination relation given by

$$\left[\frac{(1 + \frac{\eta_2}{\eta_1})zf'(z)}{zf'_{j,k}(z) + \frac{\eta_2}{\eta_1}f_{j,k}(z)} \right]^\eta \left(1 + \delta + \frac{zf''(z)}{f'(z)} - \frac{z(zf'_{j,k}(z))' + \frac{\eta_2}{\eta_1}zf'_{j,k}(z)}{zf'_{j,k}(z) + \frac{\eta_2}{\eta_1}f_{j,k}(z)} \right) \prec \delta\ell(z) + \frac{z\ell'(z)}{\eta} \quad (21)$$

holds, then $\frac{(\eta_1 + \eta_2)zf'(z)}{\eta_1zf'_{j,k}(z) + \eta_2f_{j,k}(z)} \prec [\ell(z)]^{\frac{1}{\eta}}$.

Proof. In order to acquire the appropriate proof, we examine the functional h as in (14), where $h : h(0) = 1$ is holomorphic or analytic in \mathbb{E} . From (14), we can see that

$$\left[\frac{(\eta_1 + \eta_2)zf'(z)}{\eta_1zf'_{j,k}(z) + \eta_2f_{j,k}(z)} \right]^\eta \left[1 + \frac{zf''(z)}{f'(z)} - \frac{\eta_1z(zf'_{j,k}(z))' + \eta_2zf'_{j,k}(z)}{\eta_1zf'_{j,k}(z) + \eta_2f_{j,k}(z)} \right] = \frac{zh'(z)}{\eta} \quad (22)$$

or we can write

$$\left[\frac{(\eta_1 + \eta_2)zf'(z)}{\eta_1zf'_{j,k}(z) + \eta_2f_{j,k}(z)} \right]^\eta \left[1 + \delta - \frac{\eta_1z(zf'_{j,k}(z))' + \eta_2zf'_{j,k}(z)}{\eta_1zf'_{j,k}(z) + \eta_2f_{j,k}(z)} \right] = \delta h(z) + \frac{1}{\eta}zh'(z).$$

On application of (21), we can also write

$$\delta h(z) + \frac{1}{\eta}zh'(z) \prec \delta\ell(z) + \frac{1}{\eta}z\ell'(z).$$

We let $\theta(w) = \delta w$ and $\psi(w) = \frac{1}{\eta}$ which are holomorphic or analytic on \mathbb{C} containing $\ell(\mathbb{E})$. We also let

$$Q(z) = z\ell'(z)\psi[\ell(z)] = \frac{1}{\eta}z\ell'(z) \quad (z \in \mathbb{E})$$

and

$$p(z) = \theta[\ell(z)] + Q(z) = \delta\ell(z) + \frac{1}{\eta}z\ell'(z) \quad (z \in \mathbb{E}).$$

Since, $Q(0) = 0$ and $Q'(0) = \frac{1}{\eta}\ell'(0) \neq 0$. From the assumption given in (17), we observe that $Q \in \mathcal{S}^*$ in \mathbb{E} and also we note that

$$\Re\left(\frac{zp'(z)}{Q(z)}\right) = \Re\left(1 + \frac{\delta}{\eta} + \frac{z\ell''(z)}{\ell'(z)}\right) > 0 \quad (z \in \mathbb{E}).$$

Thus, Lemma 1 leads to the implication $h(z) \prec \ell(z)$.

Taking $\ell(z) = \frac{1-\alpha_1z}{1+\alpha_2z}$ in the above Theorem 11, where $-1 \leq \alpha_2 < \alpha_1 \leq 1, \eta_2 = 0$ and $\eta_1 = 1$, we arrived at the following conclusion:

Corollary 12. *Let $-1 \leq \alpha_2 < \alpha_1 \leq 1$ and $\eta \in \mathbb{C}^*$ and $\delta \in \mathbb{C}$ with $\max\left\{0, -\Re\left(\frac{\delta}{\eta}\right)\right\} \leq \frac{1-|\alpha_2|}{1+|\alpha_2|}$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition in (20) and also the subordination results given by*

$$\left[\frac{zf'(z)}{f'_{j,k}(z)}\right]^\eta \left[1 + \delta + \frac{zf''(z)}{f'(z)} - \frac{(zf'_{j,k}(z))'}{f'_{j,k}(z)}\right] \prec \delta \frac{1 + \alpha_1z}{1 + \alpha_2z} + \frac{1}{\eta} \frac{(\alpha_1 - \alpha_2)z}{(1 + \alpha_2z)^2}, z \in \mathbb{E}. \quad (23)$$

holds, then we obtain $\left[\frac{zf'(z)}{f'_{j,k}(z)}\right]^\eta \prec \frac{1+\alpha_1z}{1-\alpha_2z}$.

Taking $\eta_2 = \gamma = 1, \eta_1 = 0$ and $\ell(z) = \frac{1+z}{1-z}$ in the above Theorem 11, then we get the following corollary:

Corollary 13. *Assume that $\delta \in \mathbb{C}$ with $\Re(\delta) \geq 0$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition in (16), and moreover the subordination condition given by*

$$\left[\frac{zf'(z)}{f'_{j,k}(z)}\right]^\eta \left[1 + \delta - \frac{zf'_{j,k}(z)}{f'_{j,k}(z)} + \frac{zf''(z)}{f'(z)}\right] \prec \delta \frac{1+z}{1-z} + \frac{2z}{(1-z)^2}$$

holds, then we have

$$\left[\frac{zf'(z)}{f'_{j,k}(z)}\right]^\eta \prec \frac{1+z}{1-z}. \quad (24)$$

4. SUPERORDINATION RESULTS

Theorem 14. *Suppose that $\ell : \ell(0) = 1$ is convex in \mathbb{E} , and $\gamma \in \mathbb{C}$ with $\Re\left(\frac{\eta}{\gamma}\right) > 0$. Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition*

$$\frac{zf'(z)}{f'_{j,k}(z)} \in \mathcal{Q}$$

and the mapping ϕ in such a way that

$$\frac{zf'(z)}{f_{j,k}(z)} \left[\eta + \gamma \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{j,k}(z)}{f_{j,k}(z)} \right) \right] \in \mathcal{S}.$$

If the subordination result given by

$$\eta\ell(z) + \gamma z\ell'(z) \prec \frac{zf'(z)}{f_{j,k}(z)} \left[\eta + \gamma \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{j,k}(z)}{f_{j,k}(z)} \right) \right], \quad (25)$$

holds, then $\ell(z) \prec \frac{zf'(z)}{f_{j,k}(z)}$.

Proof. In order to acquire the appropriate proof, we examine the following functional:

$$h(z) = \frac{zf'(z)}{f_{j,k}(z)} \quad (z \in \mathbb{E}),$$

where h is holomorphic or analytic in \mathbb{E} with $h(0) = 1$. Based on the simple calculations of the above result, we may have

$$\eta h(z) + \gamma zh'(z) = \frac{zf'(z)}{f_{j,k}(z)} \left[\eta + \gamma \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{j,k}(z)}{f_{j,k}(z)} \right) \right].$$

This assumption (25) is similar to

$$\eta\ell(z) + \gamma z\ell'(z) \prec \eta h(z) + \gamma zh'(z).$$

The mappings $\theta : \theta(w) = \eta w$ and $\psi : \psi(w) = \gamma w$ are holomorphic or analytic in \mathbb{C} . Set $p(z) = z\ell'(z)\psi[\ell(z)] = \gamma z\ell'(z) : p(0) = 0$ and $p'(0) = \gamma\ell'(0) \neq 0$ for $\ell \in \mathcal{C}$. As a result of some calculation, we observe that $p \in \mathcal{S}^*$ in \mathbb{E} and also we note that

$$\Re \left(\frac{\theta'[\ell(z)]}{\psi[\ell(z)]} \right) = \Re \left(\frac{\eta}{\gamma} \right) > 0, z \in \mathbb{E}.$$

The above condition and our deduction from Lemma 2 along with the assumption laid down in (25) implies that $\ell(z) \prec p(z)$.

Setting $\ell(z) = \frac{1-\alpha_1 z}{1+\alpha_2 z}$ in the above Theorem 14, where $-1 \leq \alpha_2 < \alpha_1 \leq 1$, we may have

Corollary 15. *Assume a holomorphic mapping $\ell \in \mathcal{C} : \ell(0) = 1$ in \mathbb{E} , and $\gamma \in \mathbb{C}$ with $\Re \left(\frac{\eta}{\gamma} \right) > 0$. In addition, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition given by*

$$\frac{zf'(z)}{f_{j,k}(z)} \in \mathcal{Q}$$

and moreover suppose that the functional

$$\frac{zf'(z)}{f_{j,k}(z)} \left[\eta + \gamma \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{j,k}(z)}{f_{j,k}(z)} \right) \right] \in \mathcal{S}.$$

If

$$\eta \frac{1 - \alpha_1 z}{1 + \alpha_2 z} + \gamma \frac{(\alpha_1 - \alpha_2)z}{(1 + \alpha_2 z)^2} \prec \frac{zf'(z)}{f_{j,k}(z)} \left[\eta + \gamma \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{j,k}(z)}{f_{j,k}(z)} \right) \right], \quad (26)$$

then $\frac{1 - \alpha_1 z}{1 + \alpha_2 z} \prec \frac{zf'(z)}{f_{j,k}(z)}$.

Theorem 16. For the complex number $\eta \in \mathbb{C}^*$ and $\delta, \eta_1, \eta_2 \in \mathbb{C}$ with $\eta_1 + \eta_2 \neq 0$ and $\Re(\delta\eta) > 0$, suppose that $\ell \in \mathcal{C} : \ell(0) = 1$ in \mathbb{E} . Furthermore, consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition

$$\frac{(\eta_1 + \eta_2)zf'(z)}{\eta_1 zf'_{j,k}(z) + \eta_2 f_{j,k}(z)} \neq 0, z \in \mathbb{E}$$

and also assume thta

$$\left[\frac{(\eta_1 + \eta_2)zf'(z)}{\eta_1 zf'_{j,k}(z) + \eta_2 f_{j,k}(z)} \right]^\eta \in \mathcal{Q}.$$

If the mapping ϕ is given in such a way that

$$\phi(z) = \left[1 + \delta + \frac{zf''(z)}{f'(z)} - \frac{\eta_1 z(zf'_{j,k}(z))' + \eta_2 zf'_{j,k}(z)}{\eta_1 zf'_{j,k}(z) + \eta_2 f_{j,k}(z)} \right] \left[\frac{(\eta_1 + \eta_2)zf'(z)}{\eta_1 zf'_{j,k}(z) + \eta_2 f_{j,k}(z)} \right]^\eta \in \mathcal{S}, \quad (27)$$

and

$$\delta\ell(z) + \frac{1}{\eta}z\ell'(z) \prec \phi(z),$$

then, $\ell(z) \prec \left[\frac{(\eta_1 + \eta_2)zf'(z)}{\eta_1 zf'_{j,k}(z) + \eta_2 f_{j,k}(z)} \right]^\eta$.

5. SANDWICH RESULTS

Combining the results describe in the Theorem 4 with Theorem 14 yields the sandwich results seen below.

Theorem 17. Assume that $l_1 : l_1(0) = 1$ and $l_2 : l_2(0) = 1$ are convex mappings with $\gamma \in \mathbb{C}^*$ and $\delta, \eta_1, \eta_2 \in \mathbb{C} : \eta_1 + \eta_2 \neq 0$. Further consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition

$$\frac{zf'(z)}{f_{j,k}(z)} \neq 0, z \in \mathbb{E}.$$

and,

$$\frac{zf'(z)}{f_{j,k}(z)} \in \mathcal{Q}$$

If the functional ϕ defined by

$$\phi(z) = \frac{zf'(z)}{f_{j,k}(z)} \left[\eta + \gamma \left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'_{j,k}(z)}{f_{j,k}(z)} \right) \right] \in \mathcal{S} \quad (28)$$

then, we may write that

$$\gamma zl'_1(z) + \eta l_1(z) \prec \phi(z) \prec \gamma zl'_2(z) + \eta l_2(z).$$

Combining the results describe in the Theorem 11 with Theorem 16 yields the sandwich results seen below.

Theorem 18. Consider that $l_1 : l_1(0) = 1$ and $l_2 : l_2(0) = 1$ are convex mappings with $\eta \in \mathbb{C}^*$ and $\delta, \eta_1, \eta_2 \in \mathbb{C} : \eta_1 + \eta_2 \neq 0$. Further consider that a holomorphic mapping $f \in \mathcal{A}$ meets the condition

$$\frac{(\eta_2 + \eta_1)zf'(z)}{\eta_1zf'_{j,k}(z) + \eta_2f_{j,k}(z)} \neq 0, z \in \mathbb{E}.$$

and,

$$\left[\frac{(\eta_2 + \eta_1)zf'(z)}{\eta_1zf'_{j,k}(z) + \eta_2f_{j,k}(z)} \right]^\eta \in \mathcal{Q}$$

If the mapping ϕ defined by

$$\phi(z) = \left[\frac{(\eta_2 + \eta_1)zf'(z)}{\eta_1zf'_{j,k}(z) + \eta_2f_{j,k}(z)} \right]^\eta \left[1 + \delta + \frac{zf''(z)}{f'(z)} - \frac{\eta_2zf'_{j,k}(z) + \eta_1z(zf'_{j,k}(z))'}{\eta_1zf'_{j,k}(z) + \eta_2f_{j,k}(z)} \right] \quad (29)$$

is such that $\phi \in \mathcal{S}$ in \mathbb{E} , and

$$\frac{1}{\eta}zl'_1(z) + \delta l_1(z) \prec \phi(z) \prec \frac{1}{\eta}zl'_2(z) + \delta l_2(z).$$

6. CONCLUDING REMARKS

In this research, by making use of first order differential subordination and superordination conditions, we determined some previously known families of mappings associated with j, k -symmetrical points. We've also come up with some sandwich theorems based on specific assumptions about the parameters that are involved in our major findings.

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