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PROPERTIES OF CERTAIN NEW SUBCLASSES OF SOME ANALYTIC AND UNIVALENT FUNCTIONS IN THE OPEN UNIT DISK

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Abstract. The focus of the present work is to introduce and study two new classes of analytic and univalent functions, $S^*_{\alpha,n}(\gamma)$ and $\mathcal{K}_{\alpha,n}(\gamma)$ relating to analytic function $f_{\alpha,n}(z)$ given by

$$f_{\alpha,n}(z) = \frac{z}{(1-z^\alpha)^n}, \quad (\alpha > 0,\ n > 0).$$

Several fascinating properties such as coefficient inequalities, radius problem and partial sum among others for functions $F_{\alpha,n}(z)$ belonging to these new classes of analytic functions are investigated .

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1. Introduction and Definitions

Suppose that A denote the class of functions $f(z)$ given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

analytic in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ and normalized with $f(0) = 0 = 1 - f'(0)$. Also let S denote the subclass of A which are univalent in D. Further, let $S^*(\gamma)$ denote the subclass of S consisting of the functions $f(z)$ of the form (1), which are starlike of order $\gamma(0 \le \gamma < 1)$ in D. A function $f(z) \in \mathcal{K}(\gamma)$ is said to be convex of order γ $(0 \le \gamma < 1)$ in D if $f(z) \in S$ satisfies the condition that $zf'(z) \in S^*(\gamma)$, $(see[1, 3, 4, 8])$.
In view of the above definitions for classes $\mathcal{K}(\gamma)$, $S^*(\gamma)$, S and A, we conclude that

$$\mathcal{K}(\gamma) \subset S^*(\gamma) \subset S \subset A$$

and $f(z) \in S^*(\gamma)$ if and only if

$$\int_0^z \frac{f(t)}{t} dt \in \mathcal{K}(\gamma).$$

Also

$$f(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + ... \quad z \in D \tag{2}$$

is in the class $S^*(0) \equiv S^*$ while the function $f(z)$ of the form

$$f(z) = \frac{z}{1-z} = z + z^2 + z^3 + ... \quad z \in D \tag{3}$$

is in the class $\mathcal{K}(0) \equiv \mathcal{K}$. In view of (2) and (3), we can write that

$$f_\alpha(z) = \frac{z}{1-z^\alpha} = z + \sum_{k=1}^{\infty} z^{1+k\alpha} \quad (z \in D) \tag{4}$$

for some real $\alpha (0 < \alpha \leq 2)$, see [2], [5] and [6].
However, in the present work, the authors present a more generalized form of (4) whereby

$$f_{\alpha,n}(z) = \frac{z}{\big(1-z^\alpha\big)^n} = z + \sum_{k=1}^{\infty} \prod_{j=1}^{k} \Big(\frac{n+j-1}{j!}\Big) z^{1+k\alpha} \quad (z \in D) \tag{5}$$

for some real $\alpha (0 < \alpha \leq 2)$ and $n > 0$.
Further, we discuss some properties of function $f_{\alpha,n}(z)$ defined by (5) where the principal value for $z^{k\alpha}$ is considered. Using (1) and (5), we introduce a new class $A_{\alpha,n}$ of analytic functions $F_{\alpha,n}(z)$, a convolution (or Hadamard product) of $f(z)$ of the form (1) and $f_{\alpha,n}(z)$, which is usually denoted by $f(z) * f_{\alpha,n}(z)$, and with series expansion in D such that

$$F_{\alpha,n}(z) = f(z) * f_{\alpha,n}(z) = z + \sum_{k=1}^{\infty} \prod_{j=1}^{k} \Big(\frac{n+j-1}{j!}\Big) a_k z^{1+k\alpha} = f_{\alpha,n}(z) * f(z) \quad (z \in D) \tag{6}$$

for some real $\alpha (0 < \alpha \leq 2)$, $n > 0$ where the principal value for $z^{k\alpha}$ is chosen.
Now, if $F_{\alpha,n}(z) \in A_{\alpha,n}$ satisfies the condition that

$$Re\bigg(\frac{zF'_{\alpha,n}(z)}{F_{\alpha,n}}\bigg) > \gamma \quad (z \in D)$$

for some real $\gamma(0 \leq \gamma < 1)$, then we say that $F_{\alpha,n}(z) \in S^*_{\alpha,n}(\gamma)$.
Also if $F_{\alpha,n}(z) \in A_{\alpha,n}$ satisfies the condition that

$$Re\left(1 + \frac{zF''_{\alpha,n}(z)}{F'_{\alpha,n}}\right) > \gamma \quad (z \in D)$$

for some real $\gamma(0 \leq \gamma < 1)$, then we say that $F_{\alpha,n}(z) \in \mathcal{K}_{\alpha,n}(\gamma)$.
In view of the above definitions we can write that $F_{\alpha,n}(z) \in \mathcal{K}_{\alpha,n}(\gamma)$ if and only if $zF'_{\alpha,n}(z) \in S^*_{\alpha,n}(\gamma)$ and that $F_{\alpha,n}(z) \in S^*_{\alpha,n}(\gamma)$ if and only if

$$\int_0^z \frac{F_{\alpha,n}(t)}{t} dt \in \mathcal{K}_{\alpha,n}(\gamma).$$

2. Some Properties of Functions $f_{\alpha,n}(z)$

The results presented in this section include certain properties of functions $f_{\alpha,n}(z)$ and $F_{\alpha,n}(z)$ with series expansions given by (5) and (6) respectively.

Theorem 1. *Let the function $f_{\alpha,n}(z)$ be given by (5). Then $f_{\alpha,n}(z) \in S^*_{\alpha,n}\left(\frac{2-n\alpha}{2}\right)$ for $(0 < \alpha \leq 2)$, $n > 0$ and $f_{\alpha,n}(z) \in \mathcal{K}_{\alpha,n}(\gamma)$ for $(0 < \alpha < 1)$, $n > 0$.*

Proof. If $f_{\alpha,n}(z)$ is given by (5), then

$$Re\left(\frac{zf'_{\alpha,n}(z)}{f_{\alpha,n}(z)}\right) = Re\left(\frac{1 + (n\alpha - 1)z^\alpha}{1 - z^\alpha}\right)$$

$$= 1 - n\alpha + n\alpha Re\left(\frac{1}{1 - z^\alpha}\right)$$

$$= 1 - n\alpha + n\alpha Re\left(\frac{1}{1 - e^{i\alpha\theta}}\right) \quad (0 < \theta < 2\pi)$$

$$= 1 - n\alpha + \frac{n\alpha}{2} = \frac{2 - n\alpha}{2} < 1$$

which shows that $f_{\alpha,n}(z) \in S^*_{\alpha,n}\left(\frac{2-n\alpha}{2}\right)$ for $(0 < \alpha \leq 2)$ and $n > 0$.
In like manner, suppose $f_{\alpha,n}(z)$ is given by (5), then

$$Re\left(1 + \frac{zf''_{\alpha,n}(z)}{f'_{\alpha,n}(z)}\right) = Re\left(\frac{1 + \big(\alpha(n+1) - 1\big)z^\alpha}{1 - z^\alpha} + \frac{\alpha(n\alpha - 1)z^\alpha}{1 + (n\alpha - 1)z^\alpha}\right)$$

$$= \alpha(n+2) - 1 + (2 - \alpha(n+1))Re\left(\frac{1}{1 - z^\alpha}\right) - \alpha Re\left(\frac{1}{1 + (n\alpha - 1)z^\alpha}\right)$$

$$= \alpha(n+2) - 1 + (2-\alpha(n+1))Re\bigg(\frac{1}{1-e^{i\alpha\theta}}\bigg) - \alpha Re\bigg(\frac{1}{1+(n\alpha-1)e^{i\alpha\theta}}\bigg)$$

$$= \frac{\alpha(n+3)}{2} - \alpha\bigg(\frac{1+(n\alpha-1)cos\alpha\theta}{1+(n\alpha-1)^2} + 2(n\alpha-1)cos\alpha\theta\bigg) \quad (0<\theta<2\pi).$$

Letting $r = cos\alpha\theta$ and $q(r) = \frac{1+(n\alpha-1)r}{1+(n\alpha-1)^2} + 2(n\alpha-1)r$, it is trivial to show that

$$q'(r) = \frac{\alpha(n\alpha-1)(n\alpha-2)}{[1+(n\alpha-1)+2(n\alpha-1)r]^2} > 0 \quad (0<\alpha<1; n>0).$$

Therefore, we conclude that

$$Re\bigg(1+\frac{zf''(z)}{f'}\bigg) > \frac{\alpha(n+1)}{2.1} \quad \big(0<\alpha<1, n>0, z\in D\big).$$

This completes the proof of Theorem 1.

Corollary 2. *A function $f_{\alpha,n}(z)$ given by*

$$f_{\alpha,n}(z) = \frac{z}{(1-\sqrt{z})^n} \quad (n>0, z\in D)$$

belongs to the class $S^*_{\frac{1}{2},n}\Big(\frac{4-n}{4}\Big)$ and $\mathcal{K}_{\frac{1}{2},n}\Big(\frac{n+1}{4}\Big)$.

Remark 1. *In a special situation whereby $n=1$, then the above result in Corollary 2.2 immediately yield the result due to Darus and Owa [2].*

Theorem 3. *Let the function $F_{\alpha,n}(z)$ given by (6) satisfies the following inequalities*

$$\sum_{k=1}^{\infty}(k\alpha+1-\gamma)\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k| \le 1-\gamma \tag{7}$$

for some $\gamma(0\le\gamma<1)$, $n>0$ and $j\in N$. Then $F_{\alpha,n}(z)\in S^*_{\alpha,n}(\gamma)$.
The equality is attained for function $F_{\alpha,n}(z)$ given by

$$F_{\alpha,n}(z) = z + \sum_{k=1}^{\infty}\frac{(1-\gamma)e^{i\pi}}{k(k+1)(k\alpha+1-\gamma){\Pi^k}_{j=1}\Big(\frac{n+j-1}{j!}\Big)}z^{1+k\alpha} \tag{8}$$

Proof. Let the function $F_{\alpha,n}(z)$ be of the form (6). If $F_{\alpha,n}(z)$ satisfies (7), then

$$\left|\frac{zF'_{\alpha,n}(z)}{F_{\alpha,n}(z)}-1\right| = \left|\frac{\sum_{k=1}^{\infty} k\alpha\Pi_{j=1}^{k}\left(\frac{n+j-1}{j!}\right)a_k z^{k\alpha}}{1+\sum_{k=1}^{\infty}\Pi_{j=1}^{k}\left(\frac{n+j-1}{j!}\right)a_k z^{k\alpha}}\right|$$

$$\leq \frac{\sum_{k=1}^{\infty} k\alpha\Pi_{j=1}^{k}\left(\frac{n+j-1}{j!}\right)|a_k||z|^{k\alpha}}{1-\sum_{k=1}^{\infty}\Pi_{j=1}^{k}\left(\frac{n+j-1}{j!}\right)|a_k||z|^{k\alpha}} < \frac{\sum_{k=1}^{\infty} k\alpha\Pi_{j=1}^{k}\left(\frac{n+j-1}{j!}\right)|a_k|}{1-\sum_{k=1}^{\infty}\Pi_{j=1}^{k}\left(\frac{n+j-1}{j!}\right)|a_k|} \leq 1-\gamma$$

showing that $F_{\alpha,n}(z)\in S^*_{\alpha,n}(\gamma)$. Having considered $F_{\alpha,n}(z)$ of the form (6), we can see that

$$\sum_{k=1}^{\infty}(k\alpha+1-\gamma)\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k| = \sum_{k=1}^{\infty}\frac{1-\gamma}{k(k+1)}$$

$$= (1-\gamma)\sum_{k=1}^{\infty}\Big(\frac{1}{k}-\frac{1}{k+1}\Big) = 1-\gamma$$

and this completes the proof of Theorem 3.

Theorem 4. *Let the function $F_{\alpha,n}(z)$ given by (6) satisfies the following inequalities*

$$\sum_{k=1}^{\infty}(k\alpha+1)(k\alpha+1-\gamma)\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k| \leq 1-\gamma$$

for some $\gamma(0\leq\gamma<1)$, $n>0$ and $j\in N$. Then $F_{\alpha,n}(z)\in\mathcal{K}^*_{\alpha,n}(\gamma)$.
The equality is attained for function $F_{\alpha,n}(z)$ given by

$$F_{\alpha,n}(z) = z+\sum_{k=1}^{\infty}\frac{(1-\gamma)(1+k\alpha)e^{i\pi}}{k(k+1)(k\alpha+1-\gamma)\Pi^{k}{}_{j=1}\left(\frac{n+j-1}{j!}\right)}z^{1+k\alpha}$$

Theorem 5. *Suppose that $F_{\alpha,n}(z)$ is of the form (6) with $arg a_k = \pi - k\alpha\theta$ $(0 < \theta < 2\pi)$. Then $F_{\alpha,n}(z)\in S^*_{\alpha,n}(\gamma)$ if and only if*

$$\sum_{k=1}^{\infty}(k\alpha+1-\gamma)\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k| \leq 1-\gamma \tag{9}$$

for some γ $(0\leq\gamma<1)$, $n>0$ and $j\in N$. The equality holds for function $F_{\alpha,n}(z)$ given by

$$F_{\alpha,n}(z) = z+\sum_{k=1}^{\infty}\frac{(1-\gamma)e^{i(\pi-k\alpha\theta)}}{k(k+1)(k\alpha+1-\gamma)\Pi^{k}{}_{j=1}\left(\frac{n+j-1}{j!}\right)}z^{1+k\alpha}$$

Proof. Theorem 3 implies that if $F_{\alpha,n}(z)$ satisfies (9), then $F_{\alpha,n}(z) \in S^*_{\alpha,n}(\gamma)$. Next, we let $F_{\alpha,n}(z) \in S^*_{\alpha,n}(\gamma)$. Then

$$Re\Big(\frac{zF'_{\alpha,n}(z)}{F_{\alpha,n}(z)}\Big) = Re\Big(\frac{1+\sum_{k=1}^{\infty}(1+k\alpha)\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)a_k z^{k\alpha}}{1+\sum_{k=1}^{\infty}\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)a_k z^{k\alpha}}\Big).$$

If we consider $z = re^{i\theta}$, then we say that

$$Re\Big(\frac{zF'_{\alpha,n}(z)}{F_{\alpha,n}(z)}\Big) = \frac{1-\sum_{k=1}^{\infty}(1+k\alpha)\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}}{1-\sum_{k=1}^{\infty}\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}}$$

$$= 1 - \frac{\sum_{k=1}^{\infty}k\alpha\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}}{1-\sum_{k=1}^{\infty}\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}} > \gamma.$$

It implies that

$$= \frac{\sum_{k=1}^{\infty}k\alpha\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}}{1-\sum_{k=1}^{\infty}\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}} \le 1-\gamma.$$

That is

$$\sum_{k=1}^{\infty}(k\alpha+1-\gamma)\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k| \le 1-\gamma.$$

Therefore $F_{\alpha,n}(z) \in S^*_{\alpha,n}(\gamma)$ if and only if the coefficient inequality (9) holds true and this concludes the proof.

Similarly for the class $\mathcal{K}_{\alpha,n}(\gamma)$, we have the following interesting result of which the proof is similar to that of theorem 5.

Theorem 6. *Let function $F_{\alpha,n}(z)$ be of the form (6) with $arga_k = \pi - k\alpha\theta(0 < \theta < 2\pi)$. Then $F_{\alpha,n}(z) \in \mathcal{K}_{\alpha,n}(\gamma)$ if and only if*

$$\sum_{k=1}^{\infty}(k\alpha+1)(k\alpha+1-\gamma)\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k| \le 1-\gamma$$

for some γ $(0 \le \gamma < 1)$, $n > 0$ and $j \in N$. The equality holds for function $F_{\alpha,n}(z)$ given by

$$F_{\alpha,n}(z) = z + \sum_{k=1}^{\infty}\frac{(1+k\alpha)(1-\gamma)e^{i(\pi-k\alpha\theta)}}{k(k+1)(k\alpha+1-\gamma)\Pi^{k}{}_{j=1}\Big(\frac{n+j-1}{j!}\Big)}z^{1+k\alpha}.$$

For recent works on Radius problems, see [?] and [?] among others.

3. Radius problems

Here, we examine the function $g_{\alpha,n}(z)$ given by

$$g_{\alpha,n}(z) = \frac{z}{(1-z^{\alpha})^{n},} \quad (z \in D) \tag{10}$$

for some real $\alpha > 2$ and $n > 0$. Then, we say that $g_{\alpha,n}(z)$ neither belongs to class $S^{*}_{\alpha,n}(\gamma)$ nor $\mathcal{K}_{\alpha,n}(\gamma)$ for any real $\gamma(0 \leq \gamma < 1)$ and $n > 0$.
Now using analytic function defined in (10), we obtain the following results.

Theorem 7. *Let function $g_{\alpha,n}(z)$ be of the form (10) with $\alpha > 2$ and $n > 0$, then*

$$Re\Big(\frac{zg'_{\alpha,n}(z)}{g_{\alpha,n}(z)}\Big) > \frac{1-(n\alpha-1)r^{\alpha}}{1+r^{\alpha}}(0 < |z| = r < 1). \tag{11}$$

Proof. For $g_{\alpha,n}(z)$ given by (10), we can see that

$$\frac{zg'_{\alpha,n}(z)}{g_{\alpha,n}(z)} = \frac{1+(n\alpha-1)r^{\alpha}e^{i\alpha\theta}}{1-r^{\alpha}e^{i\alpha\theta}} = \frac{e^{-i\alpha\theta}+(n\alpha-1)r^{\alpha}}{e^{-i\alpha\theta}-r^{\alpha}}$$

for $z = re^{i\theta}$. This implies that

$$\frac{zg'_{\alpha,n}(z)}{g_{\alpha,n}(z)} = \frac{1+(n\alpha-2)r^{\alpha}\cos\alpha\theta-(n\alpha-1)r^{2\alpha}}{1+r^{2\alpha}-2r^{\alpha}\cos\alpha\theta}.$$

Now suppose that

$$h_{\alpha,n}(\rho) = \frac{1+(n\alpha-2)r^{\alpha}\rho-(n\alpha-1)r^{2\alpha}}{1+r^{2\alpha}-2r^{\alpha}\rho}$$

where $\rho = \rho\cos\alpha\theta$. Obviously,

$$h'_{\alpha,n}(\rho) > 0.$$

Therefore

$$Re\Big(\frac{zg'_{\alpha,n}(z)}{g_{\alpha,n}(z)}\Big) > \frac{1-(n\alpha-1)r^{\alpha}}{1+r^{\alpha}}.$$

Corollary 8. *Let function $g_{\alpha,n}(z)$ be of the form (10) with $\alpha > 2$ and $n > 0$, then*

$$Re\Big(\frac{zg'_{\alpha,n}(z)}{g_{\alpha,n}(z)}\Big) > \gamma \quad (0 \leq \gamma < 1) \tag{12}$$

for $0 < |z| \leq \left(\frac{(1-\gamma)}{\gamma+n\alpha-1}\right)^{\frac{1}{\alpha}} < 1$.

Proof. Let

$$Re\Big(\frac{zg'_{\alpha,n}(z)}{g_{\alpha,n}(z)}\Big) > \frac{1-(n\alpha-1)r^{\alpha}}{1+r^{\alpha}} \geq \gamma.$$

Then

$$0 < r \leq \left(\frac{(1-\gamma)}{\gamma+n\alpha-1}\right)^{\frac{1}{\alpha}} < 1.$$

Remark 2. *(1.) Suppose $\gamma = 0$ in (12). Then*

$$0 < |z| \leq \left(\frac{1}{n\alpha-1}\right)^{\frac{1}{\alpha}} < 1.$$

and if $\gamma = \frac{1}{2}$, then

$$0 < |z| \leq \left(\frac{1}{2n\alpha-1}\right)^{\frac{1}{\alpha}} < 1.$$

(2.) Suppose $\gamma = 0$, $n = 1$ and $\gamma = \frac{1}{2}, n = 1$ in (12). Then

$$0 < |z| \leq \left(\frac{1}{\alpha-1}\right)^{\frac{1}{\alpha}} < 1.$$

and

$$0 < |z| \leq \left(\frac{1}{2\alpha-1}\right)^{\frac{1}{\alpha}} < 1$$

respectively. This result is due to Darus and Owa [2].
(3.) Suppose $\gamma = 0$, $n = 2$ and $\gamma = \frac{1}{2}, n = 2$ in (12). Then

$$0 < |z| \leq \left(\frac{1}{2\alpha-1}\right)^{\frac{1}{\alpha}} < 1.$$

and

$$0 < |z| \leq \left(\frac{1}{4\alpha-1}\right)^{\frac{1}{\alpha}} < 1$$

respectively.

4. Partial sums

We conclude this paper by discussing the partial sums of function $F_{\alpha,n}(z)$ given by (6). So, in view of (6), we have that

$$F_{\alpha,n}(z) = z + \Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)a_k z^{1+k\alpha} \quad (k = 1, 2, 3, ... \quad and \quad j \in N) \tag{13}$$

for some real α $(0 < \alpha \leq 2)$, $n > 0$. Next is the result on partial sums.

Theorem 9. *Let $F_{\alpha,n}(z)$ be given by (13) with $|a_k| \leq 1$ and $n > 0$. Then*

$$Re\Big(\frac{zF'_{\alpha,n}(z)}{F_{\alpha,n}(z)}\Big) > \frac{1-(1+k\alpha)\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|}{1-\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|} \quad (z \in D) \tag{14}$$

and

$$Re\Big(\frac{zF'_{\alpha,n}(z)}{F_{\alpha,n}(z)}\Big) \geq \frac{1-(1+k\alpha)\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)r^{k\alpha}}{1-\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)r^{k\alpha}} \quad (|z| = r < 1). \tag{15}$$

Proof. It is trivial to see that

$$Re\Big(\frac{zF'_{\alpha,n}(z)}{F_{\alpha,n}(z)}\Big) = Re\Bigg(\frac{1+(1+k\alpha)\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)a_k z^{k\alpha}}{1+\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)a_k z^{k\alpha}}\Bigg)$$

$$= Re\Bigg(1+\frac{k\alpha\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)a_k z^{k\alpha}}{1+\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)a_k z^{k\alpha}}\Bigg)$$

$$= 1 + Re\Bigg(\frac{k\alpha\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}\big(\cos(k\alpha\theta+\psi)+i\sin(k\alpha\theta+\psi)\big)}{1+\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}\big(\cos(k\alpha\theta+\psi)+i\sin(k\alpha\theta+\psi)\big)}\Bigg)$$

where $a_k = |a_k|e^{i\theta}$ and $z = re^{i\theta}$. Then, we have that

$$Re\Big(\frac{zF'_{\alpha,n}(z)}{F_{\alpha,n}(z)}\Big) = 1 + \frac{k\alpha\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}\cos(k\alpha\theta+\psi)}{1+\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}\cos(k\alpha\theta+\psi)}$$

$$= 1 + \frac{k\alpha\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}\Big[\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}+\cos(k\alpha\theta+\psi)\Big]}{1+2\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}\cos(k\alpha\theta+\psi)+\Big[\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)\Big]^2|a_k|^2r^{2k\alpha}}.$$

Now, setting

$$h(t) = \frac{\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha} + t}{1 + 2\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}t + \Big[\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)\Big]^2|a_k|^2r^{2k\alpha}}$$

where $t = \cos(k\alpha\theta + \psi)$, then $h'(t) > 0$ with $|a_k| \leq 1$. Thus

$$Re\Big(\frac{zF'_{\alpha,n}(z)}{F_{\alpha,n}(z)}\Big) > 1 - \frac{k\alpha\Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}}{1 - \Pi_{j=1}^{k}\Big(\frac{n+j-1}{j!}\Big)|a_k|r^{k\alpha}} \quad (0 \leq r < 1). \tag{16}$$

Making $r \to 1$ in (16), we obtain (14) and letting $|a_k| = 1$ in (16), we obtain (15). For recent work on Partial Sums, interested reader can see among others, Darus and Ibrahim [1], [2]Darus and Owa [2] and Hayami *el al.* [7].

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