Acta Universitatis Apulensis No. 67/2021
ISSN: 1582-5329 pp. 111-123
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2021.67.10

NORMALIZED MULTIVALENT FUNCTIONS CONNECTED WITH GENERALIZED MITTAG-LEFFLER FUNCTIONS

Om Ahuja, Sushil Kumar, Asena Çetinkaya

Abstract. In this paper, we study geometric properties of a class of multivalent functions with negative coefficients that are connected with the generalized Mittag-Leffler functions.

2010 *Mathematics Subject Classification*: 30C45, 30C50.

Keywords: Mittag-Leffler function, modified Hadamard product, multivalent functions, starlike functions.

1. Introduction

About a century ago, a Swedish mathematician G.M. Mittag-Leffler [11] discovered a celebrated function E_α defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \quad (\alpha \in \mathbb{C}; \mathrm{Re}(\alpha) > 0),$$

where $\Gamma(.)$ denotes the Gamma function. It is observed that the Mittag-Leffler function is an entire function of z with order $[\mathrm{Re}(\alpha)]^{-1}$. In 1905, Wiman [17] studied the generalized Mittag-Leffler function $E_{\alpha,\beta}$ given by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \quad (\alpha, \beta \in \mathbb{C}; \mathrm{Re}(\alpha) > 0; \mathrm{Re}(\beta) > 0).$$

Since then, the Mittag-Leffler function and its various generalizations arose in the solution of fractional differential equations, super diffusive transport problems and so on. For some details one may refer to [8] and references therein.

More recently, Bansal and Mehrez [3] studied a new function $E^\lambda_{\alpha,\beta}$ for $0 \le \lambda \le 1$ defined by

$$E^\lambda_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)(\lambda n! + (1-\lambda))}. \tag{1}$$

It is noted that $E^{0}_{\alpha,1}(z) =: E_{\alpha}(z)$ [11], $E^{0}_{\alpha,\beta}(z) =: E_{\alpha,\beta}(z)$ [17] and $E^{1}_{\alpha,\beta}(z) =: W_{\alpha,\beta}(z)$ [18].

The functions E_{α}, $E_{\alpha,\beta}$, $W_{\alpha,\beta}$ and many of their generalizations, studied in the last two decades, lack normalizations and so they do not belong to the class $\mathcal{A}$ of all analytic and normalized functions of the form

$$f(z) = z + \sum_{n=1}^{\infty} a_{n+1} z^{n+1}$$

for all z in the open unit disc $\mathbb{D} := \{z : |z| < 1\}$. Therefore, some researchers normalized these functions and studied geometric properties, such as starlikeness, convexity, close-to-convexity and partial sums, for the functions E_{α}, $E_{\alpha,\beta}$, $W_{\alpha,\beta}$ and their generalizations. For some details, refer [2, 5, 7, 10, 12, 16].

By using a similar method, we normalize the function $E^{\lambda}_{\alpha,\beta}$ defined in (1) by letting

$$\mathbb{T}^{\lambda}_{\alpha,\beta}(z) = \Gamma(\beta) z E^{\lambda}_{\alpha,\beta}(z) = z + \sum_{n=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)(\lambda n! + (1-\lambda))} z^{n+1}.$$

We notice that the normalized function $\mathbb{T}^{\lambda}_{\alpha,\beta}$ is a natural extension of the exponential, hyperbolic and trigonometric functions, and some special cases can be given as follows; for example

$$\begin{cases} \mathbb{T}^{0}_{0,1}(z) = \frac{z}{1-z}, & \mathbb{T}^{0}_{1,2}(z) = e^{z} - 1, & \mathbb{T}^{1}_{0,1}(z) = z e^{z}, \\ \mathbb{T}^{0}_{1,1}(z) = z e^{z}, & \mathbb{T}^{0}_{2,2}(z) = \sqrt{z}\sinh(\sqrt{z}), & \mathbb{T}^{1}_{2,2}(z) = z\ {}_{0}F_{2}\left(-; 1, \frac{3}{2}; \frac{z}{4}\right), \\ \mathbb{T}^{0}_{2,1}(z) = z\cosh(\sqrt{z}), & \mathbb{T}^{0}_{2,3}(z) = 2[\cosh(\sqrt{z}) - 1], & \mathbb{T}^{1}_{2,3}(z) = z\ {}_{0}F_{2}\left(-; \frac{3}{2}, 2; \frac{z}{4}\right). \end{cases}$$

Let $\mathcal{A}_p$ be the class of all normalized analytic and $p-$valent (multivalent) functions f defined in $\mathbb{D}$ given by the power series

$$f(z) = z^{p} + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N}). \tag{2}$$

Note that $\mathcal{A}_1 =: \mathcal{A}$. If f_1 and f_2 are analytic in $\mathbb{D}$, then f_1 is said to be subordinate to f_2, written as $f_1 \prec f_2$, if there exists an analytic function w satisfying $w(0) = 0$ and $|w(z)| < 1$ such that $f_1(z) = f_2(w(z))$. We also note that if f_2 is univalent in $\mathbb{D}$, then $f_1 \prec f_2$ is equivalent to $f_1(0) = f_2(0)$ and $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$ for all $z \in \mathbb{D}$.

For $0 \leq \gamma < p$, a function f belonging to $\mathcal{A}_p$ is said to be $p-$valently starlike of order γ, $p-$valently convex of order γ, and $p-$valently close-to-convex of order γ if it holds the inequalities $\mathrm{Re}(zf'(z)/f(z)) > \gamma$, $\mathrm{Re}(1 + zf''(z)/f'(z)) > \gamma$ and $\mathrm{Re}(f'(z)/z^{p-1}) > \gamma$ for all $z \in \mathbb{D}$, respectively. The classes of such functions are denoted by $\mathcal{S}_p^*(\gamma)$, $\mathcal{K}_p(\gamma)$ and $\mathcal{C}_p(\gamma)$, respectively. It is noted that, a function $f \in \mathcal{K}_p(\gamma)$ if and only if $zf'/p \in \mathcal{S}_p^*(\gamma)$. For definitions and properties of such functions, one may refer to [1, 6, 15] and relevant references therein.

Corresponding to the function $\mathbb{T}_{\alpha,\beta}^{\lambda}$ in $\mathcal{A}$, we also define the normalized function $\mathbb{E}_{p,\alpha,\beta}^{\lambda}$ in $\mathcal{A}_p$ as follows:

$$\mathbb{E}_{p,\alpha,\beta}^{\lambda}(z) = \Gamma(\beta) z^p E_{\alpha,\beta}^{\lambda}(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)(\lambda n! + (1-\lambda))} z^{n+p}.$$

Using the function $\mathbb{E}_{p,\alpha,\beta}^{\lambda}$, the linear operator $\mathcal{M}_{p,\alpha,\beta}^{\lambda} f : \mathcal{A}_p \to \mathcal{A}_p$ is given by

$$\mathcal{M}_{p,\alpha,\beta}^{\lambda} f(z) = \mathbb{E}_{p,\alpha,\beta}^{\lambda}(z) * f(z) = z^p + \sum_{n=1}^{\infty} \frac{\Gamma(\beta)}{\Gamma(\alpha n + \beta)(\lambda n! + (1-\lambda))} a_{n+p} z^{n+p}, \quad (3)$$

where $z \in \mathbb{D}$, $0 \leq \lambda \leq 1$, $\alpha, \beta \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, and when $\mathrm{Re}(\alpha) = 0$ and $\beta \neq 0$. A function f defined by (3) is called $p-$valent lambda$-$generalized Mittag-Leffler function.

By making use of the $p-$valent linear operator $\mathcal{M}_{p,\alpha,\beta}^{\lambda}$ and the concept of subordination, we introduce a subclass $\mathcal{S}_{\alpha,\beta}^{\lambda}(p, \gamma, A, B)$ of the family $\mathcal{A}_p$.

Definition 1. *For fixed parameters A, B, γ $(-1 \leq A < B \leq 1,\ 0 \leq \gamma < p,\ p \in \mathbb{N})$ and the operator $\mathcal{M}_{p,\alpha,\beta}^{\lambda} f$ defined by (3), a function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{S}_{\alpha,\beta}^{\lambda}(p, \gamma, A, B)$ if it satisfies the condition*

$$\frac{1}{p-\gamma}\left(\frac{z(\mathcal{M}_{p,\alpha,\beta}^{\lambda} f(z))'}{\mathcal{M}_{p,\alpha,\beta}^{\lambda} f(z)} - \gamma\right) \prec \frac{1+Az}{1+Bz} \quad (z \in \mathbb{D}), \quad (4)$$

or equivalently

$$\frac{z(\mathcal{M}_{p,\alpha,\beta}^{\lambda} f(z))'}{\mathcal{M}_{p,\alpha,\beta}^{\lambda} f(z)} \prec \frac{p + \big(pB + (A-B)(p-\gamma)\big)z}{1+Bz} \quad (z \in \mathbb{D}).$$

Remark 1. *It is straighforward to verify that $f \in \mathcal{S}_{\alpha,\beta}^{\lambda}(p, \gamma, A, B)$ if and only if*

$$\left| \frac{\frac{z(\mathcal{M}_{p,\alpha,\beta}^{\lambda} f(z))'}{\mathcal{M}_{p,\alpha,\beta}^{\lambda} f(z)} - p}{B\frac{z(\mathcal{M}_{p,\alpha,\beta}^{\lambda} f(z))'}{\mathcal{M}_{p,\alpha,\beta}^{\lambda} f(z)} - [pB + (A-B)(p-\gamma)]} \right| < 1 \quad (z \in \mathbb{D}), \quad (5)$$

where $\mathcal{M}_{p,\alpha,\beta}^{\lambda} f$ is defined by (3).

By taking different values of the parameters $\alpha, \beta, \lambda, \gamma, p, A$ and B, we may obtain several known special cases of the family $\mathcal{S}_{\alpha,\beta}^{\lambda}(p, \gamma, A, B)$; for example

i) $\mathcal{S}_{0,1}^{0}(p, \gamma, A, B) \equiv \mathcal{S}_p^*(\gamma, A, B)$, (Aouf [1]).

ii) $\mathcal{S}_{0,1}^{0}(p, 0, A, B) \equiv \mathcal{S}_p^*(A, B)$, (Goel [6]).

Let $\mathcal{T}_p$ be a subclass of functions in $\mathcal{A}_p$ contsisting of functions of the form

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (a_{n+p} \geq 0; p \in \mathbb{N}). \tag{6}$$

We define the class

$$\mathcal{TS}_{\alpha,\beta}^{\lambda}(p, \gamma, A, B) := \mathcal{T}_p \cap \mathcal{S}_{\alpha,\beta}^{\lambda}(p, \gamma, A, B).$$

Obviously, $\mathcal{TS}_{\alpha,\beta}^{\lambda}(p, \gamma, A, B) \subset \mathcal{S}_{\alpha,\beta}^{\lambda}(p, \gamma, A, B)$. In addition, for the classes $\mathcal{S}_p^*(\gamma)$, $\mathcal{K}_p(\gamma)$ and $\mathcal{C}_p(\gamma)$ we have

$$\mathcal{TS}_p^*(\gamma) := \mathcal{T}_p \cap \mathcal{S}_p^*(\gamma), \quad \mathcal{TK}_p(\gamma) := \mathcal{T}_p \cap \mathcal{K}_p(\gamma), \quad \mathcal{TC}_p(\gamma) := \mathcal{T}_p \cap \mathcal{C}_p(\gamma).$$

Clearly, it is observed that

i) $\mathcal{TS}_{0,1}^{0}(p, \gamma, A, B) \equiv \mathcal{T}_p^*(\gamma, A, B)$, (Aouf [1]).

ii) $\mathcal{TS}_{0,1}^{0}(p, 0, A, B) \equiv \mathcal{T}_p^*(A, B)$, (Goel [6]).

By assigning specific values to $\alpha, \beta, \lambda, \gamma, p, A$ and B, we also get the results by Silverman [14] and Gupta and Jain [9].

In this paper, we study several geometric properties of the class $\mathcal{TS}_{\alpha,\beta}^{\lambda}(p, \gamma, A, B)$. In particular, we determine sharp coefficient inequality, growth and distortion properties, radii of $p-$valently starlikeness, convexity and close-to-convexity for the class $\mathcal{TS}_{\alpha,\beta}^{\lambda}(p, \gamma, A, B)$. Finally, we obatin some results related to modified Hadamard products of the functions belonging to this class.

2. Main Results

First result of this section provides a necessary and sufficient condition for the function of the form (6) to be in the class $\mathcal{TS}_{\alpha,\beta}^{\lambda}(p, \gamma, A, B)$.

Theorem 1. *Let a function f be given by* (6). *Then $f \in \mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\gamma,A,B)$ if and only if*

$$\sum_{n=1}^{\infty}((1+B)n+(B-A)(p-\gamma))\psi_n|a_{n+p}| \leq (B-A)(p-\gamma), \tag{7}$$

where

$$\psi_n = \frac{\Gamma(\beta)}{\Gamma(\alpha n+\beta)(\lambda n!+(1-\lambda))}. \tag{8}$$

The result is sharp.

Proof. Suppose that the inequality (7) holds true, then using (5) and (6), we get

$$\begin{aligned}
&\left|z(\mathcal{M}^{\lambda}_{p,\alpha,\beta}f(z))' - p\mathcal{M}^{\lambda}_{p,\alpha,\beta}f(z)\right| \\
&\qquad - \left|Bz(\mathcal{M}^{\lambda}_{p,\alpha,\beta}f(z))' - [(A-B)(p-\gamma)+pB]\mathcal{M}^{\lambda}_{p,\alpha,\beta}f(z)\right| \\
&= \left|\sum_{n=1}^{\infty} -n\psi_n a_{n+p}z^{n+p}\right| - \left|(B-A)(p-\gamma)z^p - \sum_{n=1}^{\infty}(Bn-(A-B)(p-\gamma))\psi_n a_{n+p}z^{n+p}\right| \\
&\leq -(B-A)(p-\gamma) + \sum_{n=1}^{\infty}((1+B)n+(B-A)(p-\gamma))\psi_n\,|a_{n+p}| \leq 0.
\end{aligned}$$

Then, by the maximum modulus theorem, we observe that $f \in \mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\gamma,A,B)$.

Conversely, if $f \in \mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\gamma,A,B)$ is given by (6), then we have

$$\begin{aligned}
&\left|\frac{\frac{z(\mathcal{M}^{\lambda}_{p,\alpha,\beta}f(z))'}{\mathcal{M}^{\lambda}_{p,\alpha,\beta}f(z)} - p}{B\frac{z(\mathcal{M}^{\lambda}_{p,\alpha,\beta}f(z))'}{\mathcal{M}^{\lambda}_{p,\alpha,\beta}f(z)} - [pB+(A-B)(p-\gamma)]}\right| \\
&\qquad = \left|\frac{\sum_{n=1}^{\infty} n\psi_n\,|a_{n+p}|\,z^{n+p}}{(B-A)(p-\gamma)z^p - \sum_{n=1}^{\infty}(Bn+(B-A)(p-\gamma))\psi_n\,|a_{n+p}|\,z^{n+p})}\right| < 1.
\end{aligned}$$

Since $\mathrm{Re}(z) \leq |z|$ for all z, then we obtain

$$\mathrm{Re}\left\{\frac{\sum_{n=1}^{\infty} n\psi_n\,|a_{n+p}|\,z^{n+p}}{(B-A)(p-\gamma)z^p - \sum_{n=1}^{\infty}(Bn+(B-A)(p-\gamma))\psi_n\,|a_{n+p}|\,z^{n+p}}\right\} < 1.$$

Choosing z to be real and letting $z \to 1^-$, we get

$$\sum_{n=1}^{\infty}((1+B)n+(B-A)(p-\gamma))\psi_n|a_{n+p}| \leq (B-A)(p-\gamma).$$

This completes the proof.

Corollary 2. *If the function f given by* (6) *is in the class $\mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\gamma,A,B)$, then*

$$|a_{n+p}| \leq \frac{(B-A)(p-\gamma)}{((1+B)n+(B-A)(p-\gamma))\psi_n},$$

where ψ_n is given by (8). *The result is sharp for the function*

$$f(z) = z^p - \frac{(B-A)(p-\gamma)}{((1+B)n+(B-A)(p-\gamma))\psi_n} z^{n+p} \qquad (n \geq 1). \tag{9}$$

In the next theorem, we obtain growth and distortion estimates for the class $\mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\gamma,A,B)$.

Theorem 3. *If a function f given by* (6) *is in the class $\mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\gamma,A,B)$, then for $|z| = r < 1$ we have*

$$|f(z)| \geq r^p - \frac{(B-A)(p-\gamma)}{(1+B+(B-A)(p-\gamma))\psi_1} r^{1+p},$$

$$|f(z)| \leq r^p + \frac{(B-A)(p-\gamma)}{(1+B+(B-A)(p-\gamma))\psi_1} r^{1+p},$$

where $\psi_1 = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. The results are sharp.

Proof. By using Theorem 1, we have

$$\begin{aligned}(1+B+(B-A)(p-\gamma))\psi_1 \sum_{n=1}^{\infty} |a_{n+p}| \\ &\leq \sum_{n=1}^{\infty}((1+B)n+(B-A)(p-\gamma))\psi_n|a_{n+p}| \\ &\leq (B-A)(p-\gamma),\end{aligned}$$

which implies that

$$\sum_{n=1}^{\infty} |a_{n+p}| \leq \frac{(B-A)(p-\gamma)}{(1+B+(B-A)(p-\gamma))\psi_1}. \tag{10}$$

Thus, by using (10) we get

$$\begin{aligned}|f(z)| &\leq |z|^p + \sum_{n=1}^{\infty} |a_{n+p}||z|^{n+p} \\ &\leq r^p + r^{1+p}\sum_{n=1}^{\infty} |a_{n+p}| \\ &\leq r^p + \frac{(B-A)(p-\gamma)}{(1+B+(B-A)(p-\gamma))\psi_1} r^{1+p}. \end{aligned} \tag{11}$$

Similarly, we obtain

$$\begin{aligned} |f(z)| &\geq |z|^p - \sum_{n=1}^{\infty} |a_{n+p}||z|^{n+p} \\ &\geq r^p - r^{1+p} \sum_{n=1}^{\infty} |a_{n+p}| \\ &\geq r^p - \frac{(B-A)(p-\gamma)}{(1+B+(B-A)(p-\gamma))\psi_1} r^{1+p}. \end{aligned} \tag{12}$$

In view of the inequalities (11) and (12), we get the desired results which are sharp for the function

$$f(z) = z^p - \frac{(B-A)(p-\gamma)}{(1+B+(B-A)(p-\gamma))\psi_1} z^{1+p}. \tag{13}$$

Theorem 4. *If a function f given by* (6) *is in the class $\mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\gamma,A,B)$, then for $|z| = r < 1$ we have*

$$|f'(z)| \geq pr^{p-1} - \frac{(B-A)(p-\gamma)(1+p)}{(1+B+(B-A)(p-\gamma))\psi_1} r^p,$$

$$|f'(z)| \leq pr^{p-1} + \frac{(B-A)(p-\gamma)(1+p)}{(1+B+(B-A)(p-\gamma))\psi_1} r^p,$$

where $\psi_1 = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. The estimates are best possible for the function (13).

Proof. We omit the proof because it is similar to the proof of Theorem 3.

Next, we compute radius of $p-$valent starlikeness, radius of $p-$valent convexity and radius of $p-$valent close-to-convexity for functions belonging to the class $\mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\gamma,A,B)$.

Theorem 5. *Let $0 \leq \gamma < p$ and let a function f defined by (6) be in the class $\mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\gamma,A,B)$. Then the function f is*

(i) $p-$valently starlike of order γ in $|z| < r_1$, where

$$r_1 = \inf_{n\geq 1} \left\{ \frac{((1+B)n + (B-A)(p-\gamma))\psi_n}{(B-A)(p-\gamma)} \times \left(\frac{p-\gamma}{n+p-\gamma}\right) \right\}^{1/n}, \tag{14}$$

(ii) $p-$valently convex of order γ in $|z|<r_2$, where

$$r_2=\inf_{n\geq 1}\left\{\frac{((1+B)n+(B-A)(p-\gamma))\psi_n}{(B-A)(p-\gamma)}\times\left(\frac{p(p-\gamma)}{(n+p)(n+p-\gamma)}\right)\right\}^{1/n} \tag{15}$$

(iii) $p-$valently close-to-convex of order γ in $|z|<r_3$, where

$$r_3=\inf_{n\geq 1}\left\{\frac{((1+B)n+(B-A)(p-\gamma))\psi_n}{(B-A)(p-\gamma)}\times\left(\frac{p-\gamma}{n+p}\right)\right\}^{1/n}. \tag{16}$$

The radius estimates are sharp for the function given by (9).

Proof. i) In view of the class $\mathcal{TS}_p^*(\gamma)$, it suffices to show that

$$\left|\frac{zf'(z)}{f(z)}-p\right|\leq p-\gamma \quad (|z|<r_1). \tag{17}$$

However, we have

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}-p\right| &= \left|\frac{-\sum_{n=1}^{\infty} n|a_{n+p}|z^{n+p}}{z^p-\sum_{n=1}^{\infty} a_{n+p}z^{n+p}}\right| \\ &\leq \frac{\sum_{n=1}^{\infty} na_{n+p}|z|^n}{1-\sum_{n=1}^{\infty}|a_{n+p}||z|^n}.\end{aligned}$$

Therefore the inequality (17) is true provided

$$\sum_{n=1}^{\infty}\frac{n+p-\gamma}{p-\gamma}|a_{n+p}||z|^n\leq 1.$$

Using (7), we obtain

$$\left(\frac{n+p-\gamma}{p-\gamma}\right)|z|^n\leq\frac{((1+B)n+(B-A)(p-\gamma))\psi_n}{(B-A)(p-\gamma)},$$

or equivalently

$$|z|\leq\left\{\frac{((1+B)n+(B-A)(p-\gamma))\psi_n}{(B-A)(p-\gamma)}\times\left(\frac{p-\gamma}{n+p-\gamma}\right)\right\}^{1/n},$$

which yields the desired radius estimate given by (14).

ii) To prove (15), it suffices to show that the class $\mathcal{TK}_p(\gamma)$ is equivalent to

$$\left|\left(1+\frac{zf''(z)}{f'(z)}\right)-p\right|\leq p-\gamma \quad (|z|<r_2),$$

and the result follows by using steps similar to part (i).

iii) To prove (16), it is sufficient to demonstrate that the class $\mathcal{TC}_p(\gamma)$ is equivalent to

$$\left|\frac{f'(z)}{z^{p-1}}-p\right|\leq p-\gamma \quad (|z|<r_3),$$

and similar steps given in part (i) leads to the required radius estimate.

Let the functions f_i $(i=1,2)$ be defined by

$$f_i(z)=z^p-\sum_{n=1}^{\infty}a_{n+p,i}z^{n+p}. \tag{18}$$

Then the modified Hadamard product of the functions f_1 and f_2 is defined by

$$(f_1*f_2)(z)=z^p-\sum_{n=1}^{\infty}a_{n+p,1}a_{n+p,2}z^{n+p}.$$

By using the definition of modified Hadamard product, we get the following theorem.

Theorem 6. *Let the functions f_i $(i=1,2)$ defined by* (18) *be in the class $\mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\gamma,A,B)$. Then $f_1*f_2\in\mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\gamma_1,A,B)$, where*

$$\gamma_1=p-\frac{(1+B)(B-A)(p-\gamma)^2\psi_1}{\left[(1+B+(B-A)(p-\gamma))\psi_1\right]^2-(B-A)^2(p-\gamma)^2\psi_1}. \tag{19}$$

Proof. In view of a method given in [13], we need to show that the largest γ_1 such that

$$\sum_{n=1}^{\infty}\frac{((1+B)n+(B-A)(p-\gamma_1))\psi_n}{(B-A)(p-\gamma_1)}a_{n+p,1}a_{n+p,2}\leq 1. \tag{20}$$

Since $f_i\in\mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\gamma,A,B)$, we have

$$\sum_{n=1}^{\infty}\frac{((1+B)n+(B-A)(p-\gamma))\psi_n}{(B-A)(p-\gamma)}a_{n+p,i}\leq 1,\ (i=1,2).$$

Using Cauchy-Schwarz inequality, we obtain

$$\sum_{n=1}^{\infty}\frac{((1+B)n+(B-A)(p-\gamma))\psi_n}{(B-A)(p-\gamma)}\sqrt{a_{n+p,1}a_{n+p,2}}\leq 1. \tag{21}$$

In view of (20) and (21), we need to show that

$$\sqrt{a_{n+p,1}a_{n+p,2}} \leq \frac{\big[((1+B)n+(B-A)(p-\gamma))\psi_n\big](p-\gamma_1)}{\big[((1+B)n+(B-A)(p-\gamma_1))\psi_n\big](p-\gamma)}.$$

Thus, in view of (21), it is sufficient to prove that

$$\frac{(B-A)(p-\gamma)}{((1+B)n+(B-A)(p-\gamma))\psi_n} \leq \frac{\big[((1+B)n+(B-A)(p-\gamma))\psi_n\big](p-\gamma_1)}{\big[((1+B)n+(B-A)(p-\gamma_1))\psi_n\big](p-\gamma)}.$$

It follows from the last inequality that

$$\gamma_1 \leq p - \frac{n(1+B)(B-A)(p-\gamma)^2\psi_n}{\big[((1+B)n+(B-A)(p-\gamma))\psi_n\big]^2-(B-A)^2(p-\gamma)^2\psi_n}.$$

Now, define a function given by

$$H(n) = p - \frac{n(1+B)(B-A)(p-\gamma)^2\psi_n}{\big[((1+B)n+(B-A)(p-\gamma))\psi_n\big]^2-(B-A)^2(p-\gamma)^2\psi_n},\ (n \geq 1).$$

We conclude that $H(n)$ is an increasing function for all $n \in \mathbb{N}$. Letting $n = 1$ in the above inequality, we conclude that

$$\gamma_1 = H(1) = p - \frac{(1+B)(B-A)(p-\gamma)^2\psi_1}{\big[(1+B+(B-A)(p-\gamma))\psi_1\big]^2-(B-A)^2(p-\gamma)^2\psi_1},$$

where $\psi_1 = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}$, and the proof is completed.

Theorem 7. *Let the functions f_i $(i = 1, 2)$ defined by* (18) *be in the class $\mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\gamma,A,B)$. Then the function*

$$s(z) = z^p - \sum_{n=1}^{\infty}(a^2_{n+p,1}+a^2_{n+p,2})z^{n+p}$$

belongs to the class $\mathcal{TS}^{\lambda}_{\alpha,\beta}(p,\delta,A,B)$, where

$$\delta = p - \frac{2(1+B)(B-A)(p-\gamma)^2\psi_1}{\big[(1+B+(B-A)(p-\gamma))\psi_1\big]^2-2(B-A)^2(p-\gamma)^2\psi_1}. \tag{22}$$

Proof. In view of Theorem 1, we write

$$\sum_{n=1}^{\infty}\left[\frac{((1+B)n+(B-A)(p-\gamma))\psi_n}{(B-A)(p-\gamma)}\right]^2 a^2_{n+p,i}$$
$$\leq \sum_{n=1}^{\infty}\left[\frac{((1+B)n+(B-A)(p-\gamma))\psi_n}{(B-A)(p-\gamma)}a_{n+p,i}\right]^2 \leq 1,\ (i = 1, 2).$$

It follows from the above inequality that

$$\sum_{n=1}^{\infty} \frac{1}{2} \left[\frac{((1+B)n + (B-A)(p-\gamma))\psi_n}{(B-A)(p-\gamma)} \right]^2 (a_{n+p,1}^2 + a_{n+p,2}^2) \leq 1.$$

Hence, we need to find the largest δ such that

$$\frac{((1+B)n + (B-A)(p-\delta))\psi_n}{(B-A)(p-\delta)} \leq \frac{1}{2} \left[\frac{((1+B)n + (B-A)(p-\gamma))\psi_n}{(B-A)(p-\gamma)} \right]^2,$$

that is

$$\delta \leq p - \frac{2n(1+B)(B-A)(p-\gamma)^2\psi_n}{\left[((1+B)n + (B-A)(p-\gamma))\psi_n\right]^2 - 2(B-A)^2(p-\gamma)^2\psi_n}.$$

Now, define a function given by

$$G(n) = p - \frac{2n(1+B)(B-A)(p-\gamma)^2\psi_n}{\left[((1+B)n + (B-A)(p-\gamma))\psi_n\right]^2 - 2(B-A)^2(p-\gamma)^2\psi_n}, \ (n \geq 1).$$

We conclude that $G(n)$ is an increasing function for all $n \in \mathbb{N}$. Letting $n = 1$ in the above inequality, we conclude that

$$\delta = G(1) = p - \frac{2(1+B)(B-A)(p-\gamma)^2\psi_1}{\left[(1 + B + (B-A)(p-\gamma))\psi_1\right]^2 - 2(B-A)^2(p-\gamma)^2\psi_1},$$

and the proof is completed.

References

[1] M. K. Aouf, *A generalization of multivalent functions with negative coefficients*, Internat. J. Math. Math. Sci. 12, 3 (1989), 493–502.

[2] D. Bansal, J. K. Prajapat, *Certain geometric properties of the Mittag-Leffler functions*, Complex Var. Elliptic Equ. 61, 3 (2016), 338–350.

[3] D. Bansal, K. Mehrez, *On a new class of functions related with Mittag-Leffler and Wright functions and their properties*, Commun. Korean Math. Soc. 35 (2020), 1123–1132.

[4] P. L. Duren, *Univalent functions*, Springer-Verlag, New York, 1983.

[5] B. A. Frasin, *An application of an operator associated with generalized Mittag-Leffler function*, Konuralp J. Math. 7, 1 (2019), 199–202.

[6] R. M. Goel, N. S. Sohi, *Multivalent functions with negative coefficients*, Indian J. Pure. Appl. Math. 12, 7 (1981), 844–853.

[7] R. Gorenflo, Y. Luchko, F. Mainardi, *Analytical properties and applications of the Wright function*, Fract. Calc. Appl. Anal. 2, 4 (1999), 383–414.

[8] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. V. Rogosin, *Mittag-Leffler functions, related topics and applications*, Springer-Verlag, Berlin, 2014.

[9] V. P. Gupta, P. K. Jain, *Certain classes of univalent functions with negative coefficients*, Bull. Austral. Math. Soc. 14 (1976), 409–416.

[10] S. Maharana, J. K. Prajapat, D. Bansal, *Geometric properties of Wright function*, Math. Bohem. 143 (2018), 99–111.

[11] G. M. Mittag-Leffler, *Sur la nouvelle fonction* $E_\alpha(x)$, C. R. Acad. Sci. Paris 137 (1903), 554–558.

[12] J. K. Prajapat, *Certain geometric properties of the Wright function*, Integral Transforms Spec. Funct. 26, 3 (2015), 203–212.

[13] A. Schild, H. Silverman, *Convolutions of univalent functions with negative coefficients*, Ann. Univ. Mariae Curie-Skłodowska Sect. A 29 (1975), 99–107.

[14] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc. 51 (1975), 109–116.

[15] H. Silverman, E. M. Silvia, *Prestarlike functions with negative coefficients*, Internat. J. Math. Math. Sci. 2, 3 (1979), 427–439.

[16] H. M. Srivastava, B. A. Frasin, V. Pescar, *Univalence of integral operators involving Mittag-Leffler functions*, Appl. Math. Inf. Sci. 11, 3 (2017), 635–641.

[17] A. Wiman, *Über den fundamentalsatz in der teorie der functionen* $E_\alpha(x)$, Acta Math. 29 (1905), 191–201.

[18] E. M. Wright, *On the coefficients of power series having exponential singularities*, J. London Math. Soc. 8 (1933), 71–79.

Om Ahuja
Department of Mathematical Sciences, Kent State University
Ohio, 44021, U.S.A
email: *oahuja@kent.edu*

Sushil Kumar
Bharati Vidyapeeth's College of Engineering
Delhi–110063, India
email: *sushilkumar16n@gmail.com*

Asena Çetinkaya
Department of Mathematics and Computer Science
Istanbul Kültür University
Istanbul, Turkey
email: *asnfigen@hotmail.com*