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ON SOME NEW SHARP REPRESENTATION THEOREMS FOR LARGE SPACES OF SUBHARMONIC FUNCTIONS

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Abstract. We introduce new general spaces of subharmonic functions in the unit disk and prove some new sharp parametric representation results for them expanding some previously known assertions. Some related issues in upper half plane and product domains will be discussed.

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1. Introduction

This paper is devoted to some new results in spaces of subharmonic functions in the unit disk. We introduce new general spaces of subharmonic functions in the unit disk and show some new embedding theorems for them. Embedding theorems and various inequalities for various spaces of subharmonic functions in various domains is an old research area. We refer the reader, for example, to [4] and various references there. See also [1], [3], [4], [5], [8], [10], [12], [13], [15]. Some arguments from [14] are crucial for this paper.

As a result from these embedding theorems we obtain immediately complete parametric representations of these new large spaces of subharmonic functions. We will at the end of this paper also shortly discuss similar type assertions in similar type large subharmonic function spaces but in the upper halfplane.

To formulate that result, we first need some definition. Let D be the unit disk on the complex plane $\mathbb{C}$ and let $T = \{|z| = 1\}$ be the unit circle. Let $SH(D)$ be the space of all subharmonic functions in D.

Let further $u \in SH(D)$, let $u^+ = \max(u, 0)$. Then as usual put

$$T(r,u) = \frac{1}{2\pi}\int_{-\pi}^{\pi} u^+(re^{i\theta})d\theta,\ r \in (0,1) \quad \text{(Nevanlinna characteristics)}$$

Let now $\alpha \geq 0$ and let

$$SH_\alpha(D) = \{u \in SH(D) : \ T(r,u) \leq \frac{C_u}{(1-r)^\alpha}, \ 0 \leq r < 1\}.$$

For $\alpha = 0$ we have classical Privalov space and known results on parametric representation (see [1], [9] , [10]). For fixed $\xi, z \in D$, $\beta > -1$, $\xi \neq 0$ we denote by $A_\beta(z,\xi)$ the following expression

$$A_\beta(z,\xi) = \left(1 - \frac{z}{\xi}\right) \exp\left\{-\left[\frac{2(\beta+1)}{\pi}\right] \int_D \frac{(1-|t|^2)^\beta}{(1-z\bar{t})^{\beta+2}} \left(\ln\left|1-\frac{t}{\xi}\right|\right) dm_2(t)\right\}.$$

These are so-called Djrbashian factors (see [1], [8], [14]).

Consider further the following spaces of subharmonic functions in the unit disk

$$SH^p_\alpha(D) = \left\{u \in SH(D) : \ \left(\int_0^1 (1-r)^\alpha \left(\int_{-\pi}^{\pi} u^+(re^{i\theta}) d\varphi\right)^p dr\right) < \infty, \right\},$$

$0 < p < +\infty$, $\alpha > -1$.

Let $B^{1,\infty}_\alpha$ be Besov space on a unit circle T

$$B^{1,\infty}_s(T) = \left\{\psi \in L_1[-\pi,\pi] : \int_0^1 \frac{\|\triangle^2_t \psi\|_{L_1}}{(t^s)} dt < +\infty\right\},$$

where $\triangle^2_t \psi\left(e^{i\theta}\right) = \psi\left(e^{i(\theta+t)}\right) - 2\psi\left(e^{i\theta}\right) + \psi\left(e^{i(\theta-t)}\right)$, $\theta \in [-\pi,\pi]$, $t \in (0,1)$, $s \in (0,2)$.

In [8] Ohlupina showed that the $SH_\alpha(D)$ coincides with the class of functions u, so that

$$u(z) = \int_D \ln|A_\beta(z,\xi)| d\mu(\xi) + Re\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\psi(e^{i\theta})d\theta}{(1-e^{-i\theta}z)^{\beta+1}}\right\},$$

$z \in D$, $\psi \in B^{1,\infty}_{\beta-\alpha+1}$, $\beta > \alpha$, $\alpha > -1$, μ is a nonnegative Borel measure in D. Similar results sharp parametric representation theorems were obtained also in mentioned work for SH^p_α spaces of subharmonic functions in the unit disk.

Note these SH_α, SH^p_α, $\alpha > -1$, $p > 0$ spaces and similar type spaces of subharmonic functions were introduced for the first time in [8], where embeddings and various interesting properties were also provided. Note similar spaces and results were given also in $\mathbb{C}^+$ (upper half spaces) of $\mathbb{C}$. We will use $A_\beta(z,\xi)$ factors actively in this paper, and some properties of these factors that were used in [1], (see also [8]).

Throughout the paper, we write C or c (with or without lower indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities), but is independent of the functions or variables being discussed.

Let μ be positive Borel measure in D. Let now $n(r) = \mu(D_r)$, where $D_r = \{z \in \mathbb{C} : |z| < r\}$, $0 < r < 1$.

One of the corner stones of the theory of subharmonic functions is the following result of Riesz on parametric representation of subharmonic functions.

Theorem 1. *(see [6], [9]) Let $u \in SH(D)$, $u \not\equiv (-\infty)$. Then there is a unique Borel measure μ so that the following parametric representation is valid for u function*

$$u(z) = \int_{D_r} \ln \left| \frac{r(\xi - z)}{r^2 - \bar{\xi} r} \right| d\mu(\xi) + h(z),$$

where $z \in D_r$ and $h(z)$ is a harmonic function in D_r.

We call μ measure Riesz measure for u function. It is a general problem to find certain concrete conditions on μ so that $u \in \mathcal{X} \subset SH(D)$, where $\mathcal{X}$ is a certain fixed subclass of $SH(D)$. We refer, for example, to [8] for such type results.

Let

$$A = \left\{ f \in SH(D) : \sup_r T(r, u) < +\infty \right\}.$$

The following sharp parametric representation theorem is classical, (see [6], [9], [10], [16]).

Theorem 2. *The A class coincides with the space of all subharmonic functions for which*

$$u(z) = \frac{1}{2\pi} \int_D \ln \left| \frac{\xi - z}{1 - \bar{\xi} z} \right| d\mu(\xi) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2) d\varphi(\theta)}{(1 - 2r\cos(\theta - \varphi) + r^2)},$$

where μ is an arbitrary nonnegative Borel measure in the unit disk for which

$$\int_D (1 - |\xi|) d\mu(\xi) < \infty,$$

and φ is an arbitrary function of bounded variation on $[-\pi, \pi]$.

We provide in this paper similar type parametric representation theorems for some new large subharmonic function spaces in the unit disk.

Theorem 3. *(see [8]) Let*

$$R_\alpha = \left\{ u \in SH(D) : \ u(z) = \int_D \ln |A_\beta(z, \xi)| d\mu(\xi) + h(z) \right\},$$

where μ is a positive Borel measure in D, $h(z)$ is harmonic function, so that

$$\int_{-\pi}^{\pi} |h(re^{i\theta})| d\theta \le \frac{c}{(1 - r)^\alpha}, \ \beta > \alpha, \ \alpha > -1.$$

Then $u \in SH_\alpha(D)$, in other words the following embedding is valid

$$R_\alpha \subset SH_\alpha, \ \alpha > -1.$$

Let further $M_p^p(f,r) = \int_T |f(r\xi)|^p d\xi, \ 0 < p < \infty, \ 0 < r < 1.$

Theorem 4. *(see [8]) Let*

$$R_{p,\alpha} = \left\{ u \in SH(D) : \ u(z) = \int_D \ln |A_\beta(z,\xi)| d\mu(\xi) + h(z) \right\},$$

where $\beta > \beta_0$, $\beta_0 = \beta_0(\alpha)$, for large enough β_0, μ is a positive Borel measure in D, so that

$$\int_0^1 (1-r)^{\alpha+p}(n(r))^p dr < \infty,$$

and h is harmonic so that

$$\int_0^1 (1-r)^\alpha M_1^p(h,r) dr < \infty,$$

where $0 < p < \infty$. Then $u \in SH_\alpha^p$, in other words the following embedding is valid

$$R_{p,\alpha} \subset SH_\alpha^p, \ 0 < p < \infty, \ \alpha > -1.$$

First we show similar type embeddings to those we formulated in our theorems above for some new large analytic area Nevanlinna type spaces. Then we show at the end of this paper that our results are sharp using rather transparent arguments.

Various embedding theorems for various spaces of subharmonic functions in various domains can be seen in various papers of various authors. We mention, for example, [1]-[12] and refer the reader for various references which can be seen there in those papers. Our arguments sometimes are sketchy since they are based in elementary estimates.

2. Main results

Our main intention is to extend these embedding theorems 3 and 4 to large new scales of spaces of subharmonic functions in the unit disk. We in particular consider the following new large spaces of subharmonic functions.

We first introduce new large spaces of subharmonic functions in the unit disk as follows and then formulate our results extending of theorems 3, 4 to these large scales of functions.

Let further

$$(SA)^p_{\alpha,\beta}(D)=\left\{f\in SH(D):\int_0^1\sup_{0<\tau<R}T(f,r)^p(1-r)^\beta(1-R)^\alpha dR<\infty\right\},$$

$\alpha>-1$, $\beta>0$, $0<p<\infty$.

$$(SB)^{p,q}_{\alpha,\beta}(D)=\left\{f\in SH(D):\int_0^1\left(\int_0^R T(f,r)^p(1-\tau)^\beta d\tau\right)^{\frac{q}{p}}(1-R)^\alpha dR<\infty\right\},$$

$0<p<\infty$, $\beta>-1$, $\alpha>-1$.

$$\left(S\tilde{B}^p_{\alpha,\beta}\right)(D)=\left\{f\in SH(D):\sup_{0<R<1}\left(\int_0^R(T(f,r))^p(1-r)^\alpha dr\right)(1-R)^\beta<\infty\right\},$$

$\alpha>-1$, $\beta\geq 0$, $0<p<\infty$.

We denote these spaces bellow sometimes simple as X_1, X_2, X_3.

Note $S\tilde{B}^p_{\alpha,\beta}$ classes if $p=\infty$, $\beta=0$ we have classes studied recently in [8] and our theorems can be viewed as direct extensions of theorems of O. Ohlupina.

Note, some sharp results on zero sets and related problems on these type analytic spaces were obtained in recent papers [14] and [15]. In these papers subclasses of our spaces consisting of analytic functions were considered and parametric representations were provided also. We formulate now main results of this paper. As follows, they extend some results provided previously in papers [14] and [8] of Ohlupina. We use actively machinery from [8].

We will assume that u is harmonic in a $U(0)$ where $U(0)$ is a neighborhood of 0 and also $u(0)>-\infty$, though this assumption can be removed probably using regularization procedure for subharmonic functions provided in [8].

Theorem 5. *Let*

$$R^p_{\alpha,\beta}=\left\{u\in SH(D):\ u(z)=\int_D\ln|A_{\widetilde{\beta}}(z,\xi)|d\mu(\xi)+h(z),\ z\in D\right\},$$

where $\widetilde{\beta}>\beta_0$, $\beta_0=\beta_0(\alpha,\beta)$, *for large enough* β_0 *and* h *is a harmonic function, so that*

$$\int_0^1\left(\int_0^R\left(\int_T|h(\tau\xi)|d\xi\right)(1-\tau)^\alpha dm_2(\tau\xi)\right)^p(1-R)^\beta dR<\infty$$

and

$$\int_0^1 n(r)^p(1-\rho)^{(\alpha+1)p+\beta+p}d\rho<\infty.$$

Then the following embedding is valid

$$R^p_{\alpha,\beta} \subset SB^p_{\alpha,\beta},\ p \le 1,\ \alpha, \beta > -1.$$

Theorem 6. *Let*

$$\widetilde{R}^p_{\alpha,\beta} = \left\{ u \in SH(D):\ u(z) = \int_D \ln |A_{\tilde{\beta}}(z,\xi)| d\mu(\xi) + h(z),\ z \in D \right\}$$

where $\tilde{\beta} > \beta_0$, $\beta_0 = \beta_0(\alpha, \beta)$, *for large enough* β_0 *and* h *is harmonic function, so that*

$$\int_0^1 \left(\sup_{0<\tau<R} \left(\int_T |h(\tau\xi)| d\xi \right) (1-\tau)^\alpha \right)^p (1-R)^\beta dR < \infty$$

and

$$\int_0^1 (1-R)^{p(\alpha+1)+\beta} (n(R))^p dR < \infty.$$

Then the following embedding is valid

$$\widetilde{R}^p_{\alpha,\beta} \subset SA^p_{\alpha,\beta},\ p \le 1,\ \alpha > -1.$$

Theorem 7. *Let*

$$\widetilde{\widetilde{R}}^p_{\beta,\nu} = \left\{ u \in SH(D):\ u(z) = \int_D \ln |A_{\widetilde{\beta}}(z,\xi)| d\mu(\xi) + h(z),\ z \in D \right\}$$

where $\widetilde{\beta} > \beta_0$, $\beta_0 = \beta_0(\beta, \nu)$, *for large enough* β_0 *and* h *is a harmonic function, so that*

$$\sup_{0<R<1} \int_0^R \left(\int_T |h(\tau\xi)| d\xi \right)^p (1-\tau)^\nu d\tau (1-R)^\beta < \infty$$

and

$$n(r) \le c(1-r)^{\dfrac{-(\beta+\nu+p+1)}{p}},\ r \in (0,1).$$

Then the following embedding is valid

$$\widetilde{\widetilde{R}}^p_{\beta,\nu} \subset S\widetilde{B}^p_{\beta,\nu},\ \nu > -1,\ \beta \ge 0,\ 0 < p < \infty.$$

Remark 1. *These Theorems 5 - 7 extends Theorems 3 and 4 to these new spaces of subharmonic functions. Theorems 5 - 7 also serve as core of proof of Theorem 11, (see below).*

Proofs of theorems

We need some properties of $A_\beta(z,\xi)$ first.

Lemma 8. *(see [8])*

1. *Let $\xi, z \in D$, $z \neq \xi$, $\xi \neq 0$, $-1 < \beta < +\infty$. Then*

$$\lim_{|\xi|\to 0} \frac{\ln |A_\beta(z,\xi)|}{\left(2\ln \frac{1}{|\xi|}\right)} = 2.$$

2. *Let $\xi, z \in D$, $\xi \neq 0$, $\beta > -1$. Then*

$$\ln |A_\beta(z,\xi)| \leq c\left(\frac{1-|\xi|^2}{|1-\bar{\xi}z|}\right)^{\beta+2}.$$

Lemma 9. *(see [8]) Let μ be positive Borel measure in $\{z : |z| < 1\}$ the unit disk. Then we have*

$$\mu(\Delta_k) \leq \mu(D_k),$$

where Δ_k and D_k are subsets of the unit disk, $D_k = \{z : |z| < 1 - \frac{1}{2^k},\ k = 0, 1, 2, \ldots\}$, $\Delta_k = \left\{\xi : 1 - \frac{1}{2^k} \leq |\xi| \leq 1 - \frac{1}{2^{k+1}},\ k \in \mathbb{Z}_+\right\}$.

First we provide the following observation concerning subharmonic u function in the unit disk and it is Riesz measure μ. Let further $n(r) = \mu(D_r)$. We combine arguments from [8] with some known estimates from [14].

We follow some arguments from [8]. We denote by X one of our classes in our theorems. Let $u \in X \cap C^{(2)}(D)$, $u(0) > -\infty$, $\triangle u$ be a Laplacian of u function. Let further

$$n(r) = \int_0^r \int_{-\pi}^{\pi} \triangle u(re^{i\varphi})d\varphi\ tdt,\ 0 < r < 1.$$

Following arguments of [8] we have

$$I = \int_{-\pi}^{\pi}\int_0^\rho \ln\frac{\rho}{r}\triangle u\left(re^{i\theta}\right)rdrd\varphi \leq \int_{-\pi}^{\pi} u^+\left(\rho e^{i\varphi}\right)d\varphi,\ \rho \in (0,1).$$

Then (see [8])

$$I \equiv \int_0^\rho \frac{1}{r}\left(\int_0^r\int_{-\pi}^{\pi}\triangle u(te^{i\varphi})tdtd\varphi\right)dr.$$

Using the fact that $n(r) = \mu(D_r) = \int_0^r\int_{-\pi}^{\pi}\triangle u(te^{i\varphi})d\varphi tdt$ (see [8]) we have

$$\int_0^\rho \frac{n(r)}{r}dr \leq c\int_{-\pi}^{\pi} u^+(\rho e^{i\varphi})d\varphi,\ \rho \in (0,1). \tag{A}$$

We now simply note that from (A) directly we have the following inequalities.

$$\int_0^1 \left(\int_0^R (1-\tau)^\alpha \int_0^r \frac{n(u)}{u} du d\tau\right)^p (1-R)^\beta dR \le C_1 \|f\|_{X_1}$$

$$\int_0^1 \left(\sup_{0<\tau<R} \left(\int_0^r \frac{n(u)}{u} du\right)(1-r)^\nu\right)^p (1-R)^\sigma dR \le C_2 \|f\|_{X_2}$$

$$\sup_{0<R<1} \left(\int_0^R \left(\int_0^\tau \frac{n(u)}{u} du\right)^p (1-\tau)^\alpha d\tau\right)(1-R)^\beta \le C_3 \|f\|_{X_3}$$

It remains to follow arguments from [14] to get what we need. Namely we have the following estimates for each function space $(X_j)_{j=1,2,3}$.

$$\int_0^1 n(\rho)^p (1-\rho)^{(\alpha+1)p+\beta+p} d\rho < \infty \quad \text{for } X_1 \text{ function space}$$

$$\int_0^1 n(\rho)^p (1-\rho)^{p(\nu+1)+\sigma} dR < \infty \quad \text{for } X_2 \text{ function space}$$

$$n(\rho) \le \tilde{c}(1-\rho)^{-\frac{1+p+\alpha+\beta}{p}} \quad \text{for } X_3 \text{ function space}$$

For general case, $0 < p < \infty$, that is when $u \in C^2(D)$, $u(0) > -\infty$ assumption is not needed we must follow again arguments from [8].

We arrived at the following theorem.

Theorem 10. *Let $u \in X_1$ or X_2 or X_3 function space, $\rho \in (0,1)$. Then we have for μ Riesz measure of subharmonic u function*
$\int_0^1 (n(\rho))^p (1-\rho)^{(\alpha+1)p+\beta+p} d\rho < \infty$, $0 < p < \infty$, $\alpha > -1$, $\beta > -1$ *for X_1 space.*
$\int_0^1 n(\rho)^p (1-\rho)^{p(\nu+1)+\sigma} d\rho < \infty$, $0 < p < \infty$, $\nu \ge 0$, $\sigma > -1$ *for X_2 space.*
$n(\rho) \le c(1-\rho)^{-\frac{1+p+\alpha+\beta}{p}}$, $\alpha > -1$, $\beta \ge 0$, $0 < p < \infty$ *for X_3 space.*

Similar results for $SH_\alpha(D)$ and $SH_\alpha^p(D)$ spaces of subharmonic functions were obtained earlier by Ohlupina in [8].

We assume further u is subharmonic in D, $u(0) > -\infty$, and if U_0 is a ball covering zero, then u is harmonic there $A_\beta(z,\xi)$ is defined for all $z, \xi \in D$.

This assumption however can be removed via standard procedure of regularization of subharmonic functions (see [8]).

Let us return now to the proof of our Theorems 5 - 7 (new embedding theorems for our new general large spaces of subharmonic functions in the unit disk).

Let further $V_\beta(z) = \int_D \ln|A_\beta(z,\xi)|d\mu(\xi)$, $z \in D$, $u(z) = V_\beta(z) + h(z)$ and based on properties of A_β we have (see Lemma 8) for $z \in D$

$$u^+(z) \le |h(z)| + C\int_D \left(\frac{1-|\xi|^2}{|1-\bar{\xi}z|}\right)^{\beta+2} d\mu(\xi)$$

Following the arguments used in proof of Theorem 1, see [8], we arrive at the following estimate

$$\int_{-\pi}^{\pi} u^+(re^{i\varphi})d\varphi \le \int_{-\pi}^{\pi} |h(re^{i\varphi})|d\varphi + \int_{-\pi}^{\pi}\left(\int_D \left(\frac{1-|\xi|^2}{|1-\bar{\xi}z|}\right)^{\beta+2} d\mu(\xi)\right)d\varphi = I_1 + I_2;$$

From here it remains to show that $I_2(r)$, $r \in (0,1)$ function satisfies certain estimates. Namely that the following estimates are valid

$$\int_0^1 \left(\int_0^R I_2(r)(1-r)^\alpha dr\right)^p (1-R)^\beta dR < \infty; \qquad (C_1)$$

$$\sup_R \int_0^R (I_2(r)^p(1-r)^\alpha dr)(1-R)^\beta < \infty; \qquad (C_2)$$

$$\int_0^1 \left(\sup_{0<r<R}(I_2(r))(1-r)^\alpha\right)^p (1-R)^\beta dR < \infty; \qquad (C_3)$$

So this arrives at another problem to estimate $I_2(r)$ in each $X_1,\ X_2,\ X_3$, space to show further that $(C_1),\ (C_2),\ (C_3)$ are valid using condition in formulation of our theorems. We have following again same ideas from [8] the following chain of estimates.

The proof of (C_2) is very similar to arguments used in [8]. Indeed we have the following estimates.

Let $\triangle_k = \left\{\xi:\ 1-\frac{1}{2^k} \le |\xi| < 1-\frac{1}{2^{k+1}}\right\}$, $r \in \triangle_k$, then $\frac{1}{2^{k+1}} \le (1-|\xi|) < \frac{1}{2^k}$, $D = \bigcup_{k=0}^{+\infty}\triangle_k$. We have to show for (C_2) that

$$\int_0^R (I_2(r))^p(1-r)^\alpha dr \le \frac{c}{(1-R)^\beta},\ R \in (0,1).$$

Note that

$$\int_{-\pi}^{\pi}\left(\int_D \frac{(1-|\xi|^2)^{\beta+2}}{|1-\bar{\xi}z|^{\tilde{\beta}+2}}d\mu(\xi)\right)d\varphi \le (z = re^{i\varphi}) \le$$

$$\le c\sum_{k=0}^{\infty}\int_{\triangle_k}\frac{(1-|\xi|^2)^{\tilde{\beta}+2}}{(1-r|\xi|)^{\tilde{\beta}+1}}d\mu(\xi) \le \frac{c}{(1-r)^{\frac{\alpha+\beta+1}{p}}},\ r \in (0,1), \qquad \text{(see [8]).}$$

Since

$$\sum_{k=0}^{\infty}\int_{\triangle_k}\frac{(1-|\xi|^2)^{\tilde{\beta}+2}}{(1-r|\xi|)^{\tilde{\beta}+1}}d\mu(\xi)\leq$$

$$\leq c\sum_{k=0}^{n}\int_{\triangle_k}\frac{(1-|\xi|^2)^{\tilde{\beta}+2}}{(1-r|\bar{\xi}|)^{\tilde{\beta}+1}}d\mu(\xi)+c_1\sum_{k=n+1}^{\infty}\int_{\triangle_k}\frac{(1-|\xi|^2)^{\tilde{\beta}+2}}{(1-r|\xi|)^{\tilde{\beta}+1}}d\mu(\xi)=$$

$$=\widetilde{I}_1+\widetilde{I}_2,\quad |\xi|\in\left[1-\frac{1}{2^k};1-\frac{1}{2^{k+1}}\right),\ k\geq 0$$

It is easy to show

$$\widetilde{I}_1\leq\frac{c}{(1-r)^{\frac{\alpha+\beta+1}{p}}},\quad (\text{see }\ [8]),$$

and

$$\widetilde{I}_2\leq\frac{c_1}{(1-r)^{\frac{\alpha+\beta+1}{p}}},\ r\in(0,1),\quad (\text{see }\ [8]).$$

The rest is clear now. We have

$$\int_0^R (I_2(r))^p(1-r)^\alpha dr\leq\frac{c}{(1-R)^\beta},\ R\in(0,1).$$

Theorem is proved for X_3 spaces.

Let us show (C_1) and (C_3) now. First (C_1). As it was shown in [8] if $(1-r_k)=\frac{1}{2^k}$, $n(r_k)=n_k$, $r_k-r_{k-1}=\frac{1}{2^k}$, then for $\beta\tilde{>}\beta_0$ we have

$$C_1(R)=\int_0^R(1-r)^\alpha I_2(r)dr\leq\tilde{\tilde{c}}\int_0^R(1-r)^\alpha\int_0^1\frac{(1-\rho)^{\tilde{\beta}+1}}{(1-r\rho)^{\tilde{\beta}+1}}n(\rho)d\rho\leq$$

$$\leq C_1\sum_{k=1}^{\infty}\frac{n_k}{2^{k(\tilde{\beta}+2)}}\left[\int_0^R\frac{(1-r)^\alpha dr}{(1-r_kr)^{\tilde{\beta}+1}}\right]\leq$$

$$\leq C\sum_{k=1}^{\infty}\frac{n_k}{2^{k(\tilde{\beta}+2)}}\frac{1}{(1-r_kR)^{\tilde{\beta}-\alpha}}.$$

Now for $p\leq 1$ (we can easily reformulate condition in our theorem on $n(r)$ in terms of n_k)

$$\int_0^1 C_1(R)^p(1-R)^\beta dr\leq c\sum_{k=1}^{\infty}\frac{(n_k^p)(2^{-k\beta})}{2^{k(\tilde{\beta}+2)p}}\left[2^{-k\alpha p}\right]\left(2^{-k}\right)\left(2^{-k\tilde{\beta}p}\right)\leq const.$$

The proof of (C_3) is similar. We use

$$\sup_{0<r<R} \frac{(1-r)^{\alpha}}{(1-r\rho)^{\tilde{\beta}+1}} \leq \frac{1}{(1-R\rho)^{\tilde{\beta}+1-\alpha}},\ R,\rho \in (0,1),$$

(a bit general form of this) and we repeat $0<p<\infty$ case almost similarly for (C_3).

Note, indeed that

$$\int_0^1 \sup_{0<r<R} \left(I_2(r)(1-r)^{\alpha}\right)^p (1-R)^{\beta} dR \leq \widetilde{C} \int_0^1 \left(\int_{-\pi}^{\pi} \left(\int_D \frac{(1-|\xi|^2)^{\beta+2} d\mu(\xi)}{|1-\bar{\xi}z|^{\beta+2-\alpha}} \right)^p d\varphi \right) (1-R)^{\beta} dR,$$

$\alpha>0,\ \beta>-1$ and it remains to repeat arguments we provided for (C_1) and (C_2) case using the fact that for $\alpha_2>\alpha_1+1,\ \alpha_1>-1$

$$\int_0^1 (1-R)^{\alpha_1}(1-R\rho)^{-\alpha_2} dR \leq \widetilde{C}(1-\rho)^{-\alpha_2+\alpha_1+1},\ \rho \in (0,1).$$

Theorem is proved.

Remark 2. *For $p>1$ similar type argument based on Hardy's inequality leads to same conclusion. We however do not consider this case here in details refereing to [8] and leaving this case to interested readers.*

The following result in combination with previous assertions of this paper will give us at once new complete parametric representations of new large spaces of subharmonic functions. This is very similar to results of Privalov, Ohlupina and other authors in the unit disk and other domains which were obtained earlier and which were also discussed by us above.

Theorem 11. *Following embeddings are sharp:*

$$R^p_{\alpha,\beta} \subset SB^p_{\alpha,\beta},\ p\leq 1,\ \alpha,\beta>-1,$$

$$\widetilde{R}^p_{\alpha,\beta} \subset SA^p_{\alpha,\beta},\quad p\leq 1,\ \alpha>-1,\ \beta\geq 0,$$

$$\widetilde{\widetilde{R}}^p_{\beta,\nu} \subset S\widetilde{B}^p_{\beta,\nu},\ \nu>-1,\ \beta\geq 0, 0<p<\infty.$$

In other words the reverse inclusions are valid also.

Proof. Consider $u(z)-V_{\beta}(z)=h(z),\ z\in D$, this function is harmonic (see [8]). Also $u(z)\leq u^+(z)$, $V_{\beta}^- = \max(0,-V_{\beta})$,

$$h^+(z) \leq u^+(z) + V_{\beta}^-(z) \leq u^+(z) + C\int_D \left(\frac{1-|\xi|^2}{|1-\bar{\xi}z|}\right)^{\beta+2} d\mu(\xi).$$

Hence (see [8]) using Lemma 8 we have that

$$\int_{-\pi}^{\pi} |h(re^{i\varphi})| d\varphi \leq \int_{-\pi}^{\pi} u^{+}(re^{i\varphi}) d\varphi + C \int_{-\pi}^{\pi} \left(\int_{D} \left(\frac{1 - |\xi|^2}{|1 - \bar{\xi} z|} \right)^{\beta+2} d\mu(\xi) \right) d\varphi + C_1,$$

$r \in (0,1)$. The rest was shown above. The small additional condition can be removed via standard regularization procedure for subharmonic functions (see [6], [8], [16]). Hence based on Theorem 10 our Theorem is proved.

We now formulate similar type sharp parametric representation type results in upper halfplane $\mathbb{C}^+$. These results were obtained recently also by Ohlupina with the help of special a_β type products in these domains on the complex plane. Arguments are close to those provided in the unit disk. Complete analogues of our sharp parametric representation theorems in the unit disk in these simple domains in the complex plane may be probably also obtained by readers.

Let $\mathbb{C}^+ = \{z \in \mathbb{C} : Im\ z > 0\}$, when $z = x + iy$, $\alpha \in (0, \infty)$, where

$$\mathbb{C}^+_\rho = \{z \in \mathbb{C} : Im\ z > \rho\}, \ \rho > 0.$$

Let $SH(\mathbb{C}^+)$ is a space of all subharmonic functions in $\mathbb{C}^+$ so that

$$\int_0^{\infty} y^{\alpha-1} \left(\int_{-\infty}^{\infty} u^{+}(x+iy) dx \right)^p dy < \infty$$

and

$$\sup_{y > y_0} \int_{-\infty}^{\infty} u^{+}(x+iy) dx \leq C < \infty. \tag{1}$$

$$\forall y_0 > 0 \quad \lim_{y \to +\infty} \sup_{y} u(iy) \geq 0. \tag{2}$$

With the help of $a_\beta(z,\xi)$ special products (see [8]), a new parametric representation were provided, namely it was shown that $u \in SH^p_\alpha(\mathbb{C}^+)$, $p,\ \alpha \in (0,\infty)$ if and only if in $\mathbb{C}^+$, for $z \in \mathbb{C}^+$

$$u(z) = \int_{\mathbb{C}^+} \ln |a_\beta(z,\xi)| d\mu(\xi) + h(z),$$

where $h(z)$ is harmonic function in $\mathbb{C}^+$ where

$$\int_0^{\infty} y^{\alpha-1} \left(\int_{-\infty}^{\infty} h(x+iy) dx \right)^p dy < \infty,$$

$\mu(\xi)$ is a positive Borel measure in $\mathbb{C}^+$ so that

$$\int_0^\infty y^{p+\alpha-1} n^p(y) dy < \infty,$$

where $n(y) = \mu(\mathbb{C}_y^+)$.

Putting same type conditions (1) and (2) we easily can try to find similar type results for our new large spaces of subharmonic functions in the unit disk which we introduced in first section. We leave this to interested readers.

Namely consider as example spaces like

$$\int_0^\infty \left(\int_0^R y^{\alpha-1} \left(\int_{-\infty}^\infty h(x+iy)dx \right)^p dy \right)^q dR,$$

where $0 < p, q < \infty$, $\alpha > 0$, or spaces with $\sup_{0<R<\infty}$ instead of integration by $\int_0^\infty$ in $\mathbb{C}^+$.

To get sharp parametric representations for these new large spaces of subharmonic functions we must probably combine arguments taken from [8] with some lines of arguments of proofs of related results for these type large spaces of subharmonic functions in $\mathbb{C}^+$ in the unit disk from this paper.

Let $T^n = T \times \ldots \times T$, $D^n = D \times \ldots \times D$, $n \in \mathbb{N}$ and let $SH(D^n)$ be a space of all subharmonic functions in D^n (by each variable separately). Let further $u \in SH(D^n)$ and

$$T(\overrightarrow{r}, u) = \frac{1}{(2\pi)^n} \int_{T^n} u^+(re^{i\theta}) d\overrightarrow{\theta},$$

where $u^+ = \max(u, 0)$, $d\overrightarrow{\theta} = d\theta_1 \cdots d\theta_n$, $r = (r_1, \ldots, r_n)$, $re^{i\theta} = (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n})$, $r_i \in (0,1)$, $i = 1, \ldots, n$ be Nevanlinna characteristic in D^n.

We can define similar spaces using $T(\overrightarrow{r}, u)$ in D^n and pose and try to solve also similar problems in D^n product domains in $\mathbb{C}^n$.

For example to study $SH^p_{\overrightarrow{\alpha}}$ spaces

$$SH^p_{\overrightarrow{\alpha}}(D^n) = \left\{ u \in SH(D^n) : \int_0^1 \cdots \int_0^1 (1-r_1)^{\alpha_1} \cdots (1-r_n)^{\alpha_n} \left(T(\overrightarrow{r}, u)\right)^p d\overrightarrow{r} < \infty, \right\},$$

$0 < p < +\infty$, $\alpha_i > -1$, $i = 1, \ldots, n$.

In higher dimension in [2] similar parametric representation was obtained for u subharmonic function in $\mathbb{C}^n$, $u(x) \neq -\infty$, and in [7] for plurisubharmonic function in $\mathbb{C}^n$.

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