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ON STAR COLORING OF LEXICOGRAPHIC PRODUCT OF GRAPHS

Sivakami R, Kaliraj K, Vernold Vivin J

Abstract. A star coloring of a graph G is a proper vertex coloring in which every path on four vertices in G is not bicolored. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G. In this paper, we determine the star chromatic number of lexicographic product of complete graph with complete graph $K_m[K_n]$, complete graph with wheel graph $K_m[W_n]$, complete graph with path $K_m[P_n]$ and path with path $P_m[P_n]$.

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1. Introduction

All graphs in this paper are finite, simple, connected and undirected graph and we follow [2, 3, 8] for terminology and notation that are not defined here. We denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. Branko Grünbaum introduced the concept of star chromatic number in 1973. A star coloring [1, 5, 6] of a graph G is a proper vertex coloring in which every path on four vertices uses at least three distinct colors. The star chromatic number $\chi_s(G)$ of G is the least number of colors needed to star color G.

During the years star coloring of graphs has been studied extensively by several authors, for instance see [1, 4, 5].

Definition 1. *A trail is called a path if all its vertices are distinct. A closed trail whose origin and internal vertices are distinct is called a cycle.*

Definition 2. *A graph G is complete if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n.*

Definition 3. *A wheel graph is a graph formed by connecting a single universal vertex to all vertices of a cycle. It is denoted by W_n with n vertices $(n \geq 4)$.*

Definition 4. *The lexicographic product [7] $G[H]$ of graphs G and H is a graph such that the vertex set of $G \cdot H$ is the Cartesian product $V(G) \times V(H)$ and any two vertices (u, v) and (x, y) are adjacent in $G[H]$ if and only if either*

- *u is adjacent with x in G or*
- *$u = x$ and v is adjacent with y in H.*

Definition 5. *The lexicographic product $G[H]$ of disjoint graphs G and H is defined as follows:*

$$V(G[H]) = V\left(G\right) \times V\left(H\right).$$

Definition 6. *Let m and n are positive integer, We define the vertex set of complete graph K_n, wheel W_n and path P_n, as follows:*

$$\begin{aligned} V\left(K_m\right) &= \{u_i : 1 \leq i \leq m\} \\ V\left(K_n\right) &= \{v_j : 1 \leq j \leq n\} \\ V\left(W_n\right) &= \{u_1\} \cup \{u_j : 2 \leq j \leq n\} \\ V\left(P_m\right) &= \{u_i : 1 \leq i \leq m\} \\ V\left(P_n\right) &= \{v_j : 1 \leq j \leq n\}. \end{aligned}$$

In the next section, we find the exact values of the star chromatic number of lexicographic product of complete graph with complete graph $K_m[K_n]$, complete graph with wheel graph $K_m[W_n]$, complete graph with path $K_m[P_n]$ and path with path $P_m[P_n]$.

2. Main Results

2.1. Star chromatic number of $K_m[K_n]$

Theorem 1. *For any positive integers m and n, the star coloring of lexicographic product of $K_m[K_n]$ is mn.*

Proof. Let

$$V(K_m[K_n]) = \bigcup_{i=1}^{m} \{s_{i,j} : 1 \leq j \leq n\},$$

where $s_{i,j}$ are the vertices of $u_i v_j$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$.
Define a mapping,

$$\sigma : V(K_m[K_n]) \to \mathbb{N}$$

as follows:

$$\begin{aligned}\sigma(s_{i,j}) &= i, \text{ for } 1 \le i \le m, \quad j = 1;\\ \sigma(s_{i,j}) &= (j-1)m + i, \text{ for } 1 \le i \le m, \quad 2 \le j \le n.\end{aligned}$$

Thus $\chi_S(K_m[K_n]) = mn$.

Suppose to the contrary, $\chi_S(K_m[K_n]) > mn$ say $mn+1$. But the $V(K_m[K_n])$ are mn, which is a contradiction to the definition of the lexicographic product, therefore $\chi_S(K_m[K_n]) \le mn$. If there exist cliques of order mn in $V(K_m[K_n])$. Thus $\chi_S(K_m[K_n]) = mn$.

2.2. Star chromatic number of $K_m[W_n]$

Theorem 2. *For any positive integer m and n,*

$$\chi_s(K_m[W_n]) = \begin{cases} mn, & for \quad m \ge 2 \text{ and } n = 4\\ mn-1, & for \quad m \ge 2 \text{ and } n = 5\\ mn-n+5, & for \quad m \ge 2 \text{ and } n \ge 6\end{cases}$$

Proof. Let

$$V(K_m[W_n]) = \bigcup_{i=1}^{m} \{w_{i,j} : 1 \le j \le n\},$$

where $w_{i,j}$ are the vertices of $u_i v_j$ $(1 \le i \le m,\ 1 \le j \le n)$.

Case 1: When $m \ge 2$ *and* $n = 4$

Let $\{c_1, \ldots, c_{mn}\}$ be the set of distinct colors. The vertices $(w_{i,j})$ where $(1 \le i \le m,\ 1 \le j \le 4)$ can be colored with color $c_1, c_2, c_3, \ldots, c_{mn}$ respectively.

Suppose to the contrary, $\chi_s(K_m[W_n]) > (mn)$, say $(mn)+1$. But the $V(K_m[W_n])$ are (mn), which is a contradiction to the definition of the lexicographic product. Thus $\chi_s(K_m[W_n]) \le (mn)$. If the result is less than mn, then it contradicts the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (mn)$, for $m \ge 2$ *and* $n = 4$.

Case 2: When $m \ge 2$ and $n = 5$

Let $\{c_1, \ldots, c_{mn-1}\}$ be the set of distinct colors. The vertices $(w_{i,j})$ where $(1 \le i \le m-1,\ 1 \le j \le 5)$ are colored with color $\{c_1, \ldots, c_{mn-1}\}$ and the vertex $u_m v_3$ and $u_m v_5$ are given the same color that is C_{m3} color.

Suppose $\chi_s(K_m[W_n]) > (mn-1)$. Then it contradicts, the chromatic number of star coloring. So $\chi_s(K_m[W_n]) \le (mn-1)$. But it contradicts, the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (mn-1)$, for $m \geq 2$ and $n = 5$.

Case 3: When $m \geq 2$ *and* $n \geq 6$

Subcase 1: When $m = 2$ *and* $n \geq 6$

Let $\{c_1, \ldots, c_{mn-n+5}\}$ be the set of distinct colors. The vertices $(w_{1,j})$ are colored with c_{2j-1} color where $1 \leq j \leq 5$, vertices $(w_{1,j})$ are colored with c_{j+5} color where $7 \leq j \leq n$ and vertex $(w_{1,6})$ are colored with c_{10} color. Similarly the vertex $w_{2,1}$ and $w_{2,2}$ are colored with c_2 and c_4 respectively. The vertices $(w_{2,2j-1})$ are colored with c_6 color, where $2 \leq j \leq \lfloor \frac{n}{2} - 1 \rfloor$, the vertices $(w_{2,4j})$ are colored with c_8 color where $1 \leq j \leq \lceil \frac{n}{4} \rceil$ and vertices $(w_{2,4j+2})$ are colored with c_{11} color where $1 \leq j \leq \lceil \frac{n}{4} \rceil$ respectively.

Subcase 2: When $m = 3$ *and* $n \geq 6$

Let $\{c_1, \ldots, c_{mn-n+5}\}$ be the set of distinct colors. The same pattern of colors are followed till $m = 2$ and $n \geq 6$. The vertex $w_{3,j}$ is colored with the preceding vertex color $(w_{1,n}+1)^{th}$ color for $j = 1$, when $j = 2$ the vertex $w_{3,j}$ is colored with preceding vertex color $(w_{1,n+1}+1)^{th}$ color and so on.

Subcase 3: When $m \geq 4$ *and* $n \geq 6$

Let $\{c_1, \ldots, c_{mn-n+5}\}$ be the set of distinct colors. The same pattern of colors are followed till $2 \leq m \leq 3$ and $n \geq 6$. The preceding subcase is applied to the vertices $w_{i,j}$ where $2 \leq i \leq 3$. The vertices $w_{4,j}$ where $1 \leq j \leq n$ are colored with the preceding vertex color $(w_{3,n}+1)^{th}$ color and so on. As m increases the same pattern of colors are given as in $(w_{4,j})^{th}$ color.

Suppose $\chi_s(K_m[W_n]) > (mn-n+5)$. Then it contradicts, the chromatic number of star coloring. So $\chi_s(K_m[W_n]) \leq (mn-n+5)$. But it contradicts, the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (mn-n+5)$, for $m \geq 2$ *and* $n \geq 6$.

2.3. Star chromatic number of $K_m[P_n]$

Theorem 3. *For any positive integer m and n,*

$$\chi_s(K_m[P_n]) = \begin{cases} mn, & when \quad m \geq 2 \text{ and } n = 2 \\ mn-1, & when \quad m \geq 2 \text{ and } n = 3 \\ (m-1)n+3, & when \quad m \geq 2 \text{ and } n \geq 4 \end{cases}$$

Proof. Let

$$V(K_m[P_n]) = \bigcup_{i=1}^{m} \{w_{i,j} : 1 \le j \le n\},$$

where $w_{i,j}$ are the vertices of $u_i v_j$ $(1 \le i \le m, 1 \le j \le n)$.

Case 1: For $m \ge 2$ *and* $n = 2$

Let $\{c_1, \ldots, c_{mn}\}$ be the set of distinct colors. The vertices $(w_{i,j})$ where $1 \le i \le m$ *and* $1 \le j \le 2$ can be colored with the color $c_1, \ldots, c_{mn}$ respectively. Suppose to the contrary, $\chi_s(K_m[W_n]) > (mn)$ say (mn)+1. But the $V(K_m[W_n])$ are (mn), which is a contradiction to the definition of the lexicographic product. $\chi_s(K_m[W_n]) \le (mn)$. If the result is less than mn, then it contradicts the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (mn)$ for $m \ge 2$ *and* $n = 2$.

Case 2: For $m \ge 2$ *and* $n = 3$

Let $\{c_1, \ldots, c_{mn-1}\}$ be the set of distinct colors. The vertices $(w_{1,j})$ are colored with c_{2j-1} color where $1 \le j \le 3$, vertices $(w_{2,j})$ are colored with c_2 color where $1 \le j \le \lceil \frac{n}{2} \rceil$ and the vertex $(w_{2,2})$ is colored with c_4 color, the vertices $(w_{3,j})$ are colored with the vertex color $(w_{1,3}+1)^{th}$ color. Similarly the vertices $(w_{4,j})$ are colored with vertex color $(w_{3,3}+1)^{th}$ color and so on.

Suppose $\chi_s(K_m[W_n]) > (mn-1)$. Then it contradicts, the chromatic number of star coloring. So $\chi_s(K_m[W_n]) \le (mn-1)$. But it contradicts, the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (mn-1)$, for $m \ge 2$ *and* $n = 3$.

Case 3: For $m \ge 2$ *and* $n \ge 4$

Subcase 1: When $m = 2$ and $n \ge 4$.

Let $\{c_1, \ldots, c_{mn-n+3}\}$ be the set of distinct colors. The vertices $(w_{1,j})$ are colored with c_{2j-1} color, where $1 \le j \le 3$, the vertices $(w_{1,4})$ is colored with color c_6, the vertex $(w_{1,j})$ are colored with the color c_{j+3} where $5 \le j \le n$, the vertices $(w_{2,2j-1})$ are colored with the color c_2, where $1 \le j \le \lceil \frac{n}{2} \rceil$, the vertices $(w_{2,4j-2})$ are colored with color c_4 where $1 \le j \le \lceil \frac{n}{4} \rceil$, the vertices $(w_{2,4j})$ are colored with color c_7 where $1 \le j \le \lfloor \frac{n}{4} \rfloor$ respectively.

Subcase 2: When $m = 3$ and $n \ge 4$.

Let $\{c_1, \ldots, c_{mn-n+3}\}$ be the set of distinct colors. The same pattern of colors are followed till $m = 2$ *and* $n \ge 4$. The vertices $(w_{3,j})$ are colored with the preceding vertex color $(w_{1,n}+1)^{th}$ color for $j=1$, when $j=2$, the vertex $(w_{3,j})$ is colored with preceding vertex color $[(w_{1,n}+1)+1]^{th}$ color and so on..

Subcase 3: When $m \geq 4$ and $n \geq 4$.

Let $\{c_1,\dots,c_{mn-n+3}\}$ be the set of distinct colors. The same pattern of colors are followed till $2 \leq m \leq 3$ *and* $n \geq 4$. The vertices $(w_{4,j})$, $1 \leq j \leq n$ are colored with the preceding vertex color $(w_{3,n}+1)^{th}$ color and so on.. As 'm' increases the same pattern of colors are given as in $(w_{4,j})^{th}$ color.

Suppose $\chi_s(K_m[W_n]) > (mn-n+3)$. Then it contradicts, the chromatic number of star coloring. So $\chi_s(K_m[W_n]) \leq (mn-n+3)$. But it contradicts, the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (mn-n+3)$, for $m \geq 2$ *and* $n \geq 4$.

2.4. Star chromatic number of $K_m[P_n]$

Theorem 4. *For any positive integer m and n,*

$$\chi_s(K_m[P_n]) = \begin{cases} mn, & when \quad m \geq 2 \text{ and } n = 2 \\ mn-1, & when \quad m \geq 2 \text{ and } n = 3 \\ (m-1)n+3, & when \quad m \geq 2 \text{ and } n \geq 4 \end{cases}$$

Proof. Let

$$V(K_m[P_n]) = \bigcup_{i=1}^{m} \{w_{i,j} : 1 \leq j \leq n\},$$

where $w_{i,j}$ are the vertices of $u_i v_j$ $(1 \leq i \leq m, 1 \leq j \leq n)$.

Case 1: For $m \geq 2$ *and* $n = 2$

Let $\{c_1,\dots,c_{mn}\}$ be the set of distinct colors. The vertices $(w_{i,j})$ where $1 \leq i \leq m$ *and* $1 \leq j \leq 2$ can be colored with the color $c_1,\dots,c_{mn}$ respectively. Suppose to the contrary, $\chi_s(K_m[W_n]) > (mn)$ say (mn)+1. But the $V(K_m[W_n])$ are (mn), which is a contradiction to the definition of the lexicographic product. $\chi_s(K_m[W_n]) \leq (mn)$. If the result is less than mn, then it contradicts the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (mn)$ for $m \geq 2$ *and* $n = 2$.

Case 2: For $m \geq 2$ *and* $n = 3$

Let $\{c_1,\dots,c_{mn-1}\}$ be the set of distinct colors. The vertices $(w_{1,j})$ are colored with c_{2j-1} color where $1 \leq j \leq 3$, vertices $(w_{2,j})$ are colored with c_2 color

where $1 \leq j \leq \lceil \frac{n}{2} \rceil$ and the vertex $(w_{2,2})$ is colored with c_4 color, the vertices $(w_{3,j})$ are colored with the vertex color $(w_{1,3}+1)^{th}$ color. Similarly the vertices $(w_{4,j})$ are colored with vertex color $(w_{3,3}+1)^{th}$ color and so on.
Suppose $\chi_s(K_m[W_n]) > (mn-1)$. Then it contradicts, the chromatic number of star coloring. So $\chi_s(K_m[W_n]) \leq (mn-1)$. But it contradicts, the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (mn-1)$, for $m \geq 2$ *and* $n = 3$.

Case 3: For $m \geq 2$ *and* $n \geq 4$

Subcase 1: When $m = 2$ and $n \geq 4$.
Let $\{c_1, \ldots, c_{mn-n+3}\}$ be the set of distinct colors. The vertices $(w_{1,j})$ are colored with c_{2j-1} color, where $1 \leq j \leq 3$, the vertices $(w_{1,4})$ is colored with color c_6, the vertex $(w_{1,j})$ are colored with the color c_{j+3} where $5 \leq j \leq n$, the vertices $(w_{2,2j-1})$ are colored with the color c_2, where $1 \leq j \leq \lceil \frac{n}{2} \rceil$, the vertices $(w_{2,4j-2})$ are colored with color c_4 where $1 \leq j \leq \lceil \frac{n}{4} \rceil$, the vertices $(w_{2,4j})$ are colored with color c_7 where $1 \leq j \leq \lfloor \frac{n}{4} \rfloor$ respectively.

Subcase 2: When $m = 3$ and $n \geq 4$.
Let $\{c_1, \ldots, c_{mn-n+3}\}$ be the set of distinct colors. The same pattern of colors are followed till $m = 2$ *and* $n \geq 4$. The vertices $(w_{3,j})$ are colored with the preceding vertex color $(w_{1,n}+1)^{th}$ color for $j = 1$, when $j = 2$, the vertex $(w_{3,j})$ is colored with preceding vertex color $[(w_{1,n}+1)+1]^{th}$ color and so on..

Subcase 3: When $m \geq 4$ and $n \geq 4$.
Let $\{c_1, \ldots, c_{mn-n+3}\}$ be the set of distinct colors. The same pattern of colors are followed till $2 \leq m \leq 3$ *and* $n \geq 4$. The vertices $(w_{4,j})$, $1 \leq j \leq n$ are colored with the preceding vertex color $(w_{3,n}+1)^{th}$ color and so on.. As 'm' increases the same pattern of colors are given as in $(w_{4,j})^{th}$ color.

Suppose $\chi_s(K_m[W_n]) > (mn-n+3)$. Then it contradicts, the chromatic number of star coloring. So $\chi_s(K_m[W_n]) \leq (mn-n+3)$. But it contradicts, the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (mn-n+3)$, for $m \geq 2$ *and* $n \geq 4$.

2.5. Star chromatic number of $P_m[P_n]$

Theorem 5. *For any positive integer m and n,*

$$\chi_s(P_m[P_n]) = \begin{cases} m+n, & for \quad m=2 \; and \; n=2,3 \\ m+n-1, & for \quad m=3 \; and \; n=2,3 \\ m+n, & for \quad m=3 \; and \; n \geq 4 \\ m+n+1, & for \quad m=2 \; and \; n \geq 4 \\ 2n+2, & for \quad m \geq 4 \; and \; n=2,3 \\ 2n+3, & for \quad m,n \geq 4 \end{cases}$$

Proof. Let

$$V(P_m[P_n]) = \bigcup_{i=1}^{m} \{w_{i,j} : 1 \leq j \leq n\},$$

where $w_{i,j}$ are the vertices of $u_i v_j$ $(1 \leq i \leq m,\ 1 \leq j \leq n)$.

Case 1: When $m = 2$ *and* $2 \leq n \leq 3$

Let $\{c_1, \ldots, c_{m+n}\}$ be the set of distinct colors. The vertices $(w_{1,j})$ where $(1 \leq j \leq 3)$ can be colored with color c_{2j-1}, the vertices $(w_{2,j})$ are colored with color c_2 where $1 \leq j \leq \lceil \frac{n}{2} \rceil$ and the vertex $(w_{2,2})$ is colored with c_4 color. Suppose $\chi_s(K_m[W_n]) > (m+n)$. Then it contradicts, the chromatic number of star coloring. So $\chi_s(K_m[W_n]) \leq (m+n)$. But it contradicts, the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (m+n)$, for $m = 2$ *and* $2 \leq n \leq 3$.

Case 2: When $m = 3$ *and* $2 \leq n \leq 3$

Let $\{c_1, \ldots, c_{m+n-1}\}$ be the set of distinct colors. The vertices $(w_{2i-1,2j-1})$ are colored with color c_1, where $1 \leq i \leq \lceil \frac{m}{2} \rceil$ & $1 \leq j \leq \lceil \frac{n}{2} \rceil$, the vertices $(w_{2i-1,2})$ are colored with color c_3 where $1 \leq i \leq \lceil \frac{m}{2} \rceil$, the vertex $(w_{2,2})$ is colored with color c_4 and the vertices $(w_{2,2j-1})$ are colored with color c_2 where $1 \leq j \leq \lceil \frac{n}{2} \rceil$.

Suppose $\chi_s(K_m[W_n]) > (m+n-1)$. Then it contradicts, the chromatic number of star coloring. So $\chi_s(K_m[W_n]) \leq (m+n-1)$. But it contradicts, the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (m+n-1)$, for $m = 3$ *and* $2 \leq n \leq 3$.

Case 3: When $m = 2$ *and* $n \geq 4$

Let $\{c_1, \ldots, c_{m+n+1}\}$ be the set of distinct colors. The vertices $(w_{1,j})$ are colored with color c_{2j-1}, where $1 \leq j \leq 3$, the vertex $(w_{1,4})$ is colored with color

c_6, the vertices $(w_{1,j})$ are colored with color c_{j+3} where $5 \leq j \leq n$, the vertices $(w_{2,2j-1})$ are colored with color c_2 where $1 \leq j \leq \lceil \frac{n}{2} \rceil$, the vertices $(w_{2,4j-2})$ are colored with color c_4 where $1 \leq j \leq \lceil \frac{n}{4} \rceil$ and the vertices $(w_{2,4j})$ are colored with color c_7 where $1 \leq j \leq \lfloor \frac{n}{4} \rfloor$.
Suppose $\chi_s(K_m[W_n]) > (m+n+1)$. Then it contradicts, the chromatic number of star coloring. So $\chi_s(K_m[W_n]) \leq (m+n+1)$. But it contradicts, the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (m+n+1)$, for $m = 2$ *and* $n \geq 4$.

Case 4: When $m = 3$ *and* $n \geq 4$
Let $\{c_1, \ldots, c_{m+n}\}$ be the set of distinct colors. The vertices $(w_{i,4j-3})$ are colored with color c_1, where $1 \leq i \leq 3$, $1 \leq j \leq \lfloor \frac{n+3}{4} \rfloor$, also the vertices $(w_{i,4j-1})$ are colored with color c_1, where $1 \leq i \leq 3$, $1 \leq j \leq \lfloor \frac{n+1}{4} \rfloor$. The vertices $(w_{i,4j-2})$ are colored with color c_3, where $1 \leq i \leq 3$, $1 \leq j \leq \lfloor \frac{n+2}{4} \rfloor$. The vertices $(w_{i,4j})$ are colored with color c_6 where $1 \leq i \leq 3$, $1 \leq j \leq \lfloor \frac{n}{4} \rfloor$. The vertex $(w_{2,1})$ is colored with color c_2, the vertex $(w_{2,2})$ is colored with color c_4, the vertex $(w_{2,3})$ is colored with color c_5. The vertices $(w_{2,j})$ are colored with color c_{j+3} where $4 \leq j \leq n$.
Suppose $\chi_s(K_m[W_n]) > (m+n)$. Then it contradicts, the chromatic number of star coloring. So $\chi_s(K_m[W_n]) \leq (m+n)$. But it contradicts, the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (m+n)$, for $m = 3$ *and* $n \geq 4$.

Case 5: When $m \geq 4$ *and* $n = 2, 3$

Subcase 1: When $m \geq 4$ *and* $n = 2$
Let $\{c_1, \ldots, c_{2n+2}\}$ be the set of distinct colors. The vertices $(w_{4i-3,1})$ and $(w_{4i-1,1})$ are colored with color c_1, where $1 \leq i \leq \lfloor \frac{m+3}{4} \rfloor$ and $1 \leq i \leq \lceil \frac{m+1}{4} \rceil$. The vertex $(w_{4i-2,1})$ is colored with color c_2, where $1 \leq i \leq \lfloor \frac{m+2}{4} \rfloor$. The vertex $(w_{4i,1})$ is colored with color c_5, where $1 \leq i \leq \lfloor \frac{m}{4} \rfloor$. The vertices $(w_{4i-3,2})$ and $(w_{4i-1,2})$ are colored with color c_3, where $1 \leq i \leq \lfloor \frac{m+3}{4} \rfloor$ and $1 \leq i \leq \lfloor \frac{m+1}{4} \rfloor$. The vertices $(w_{4i-2,2})$ are colored with color c_4, where $1 \leq i \leq \lfloor \frac{m+2}{4} \rfloor$ and the vertex $(w_{4i,1})$ is colored with color c_6 where $1 \leq i \leq \lfloor \frac{m}{4} \rfloor$.

Subcase 2: When $m \geq 4$ *and* $n = 3$
Let $\{c_1, \ldots, c_{2n+2}\}$ be the set of distinct colors. The preceding color pattern is applied till $m \geq 4$ and $1 \leq n \leq 2$. The remaining colors are given in such a way that the vertices $(w_{4i-3,3})$ and $(w_{4i-1,3})$ are colored with color c_1, where $1 \leq i \leq \lfloor \frac{m+3}{4} \rfloor$ and $1 \leq i \leq \lceil \frac{m+1}{4} \rceil$. The vertex $(w_{4i-2,3})$ is colored with color

c_7, where $1 \leq i \leq \lfloor \frac{m+2}{4} \rfloor$ and the vertex $(w_{4i,1})$ is colored with color c_8, where $1 \leq i \leq \lfloor \frac{m}{4} \rfloor$.
Suppose $\chi_s(K_m[W_n]) > (2n+2)$. Then it contradicts, the chromatic number of star coloring. So $\chi_s(K_m[W_n]) \leq (2n+2)$. But it contradicts, the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (2n+2)$, for $m \geq 4$ *and* $n = 2, 3$.

Case 6: When m *and* $n \geq 4$
Let $\{c_1, \ldots, c_{2n+3}\}$ be the set of distinct colors. The vertices $(w_{2i-1,2j-1})$ are colored with the color c_1, where $1 \leq i \leq \lfloor \frac{m+1}{2} \rfloor$ *and* $1 \leq j \leq \lfloor \frac{n+1}{2} \rfloor$, the vertices $(w_{2i-1,4j-2})$ are colored with the color c_3, where $1 \leq i \leq \lfloor \frac{m+1}{2} \rfloor$ *and* $1 \leq j \leq \lfloor \frac{n+2}{4} \rfloor$, the vertices $(w_{2i-1,4j})$ are colored with the color c_9, where $1 \leq i \leq \lfloor \frac{m+1}{2} \rfloor$ *and* $1 \leq j \leq \lfloor \frac{n}{4} \rfloor$, the vertices $(w_{4i-2,1})$ are colored with the color c_2, where $1 \leq i \leq \lfloor \frac{m+2}{4} \rfloor$. The vertices $(w_{4i-2,2})$ are colored with the color c_4, where $1 \leq i \leq \lfloor \frac{m+2}{4} \rfloor$. The vertices $(w_{4i-2,3})$ are colored with the color c_7, where $1 \leq i \leq \lfloor \frac{m+2}{4} \rfloor$. The vertices $(w_{4i-2,4})$ are colored with the color c_{10}, where $1 \leq i \leq \lfloor \frac{m+2}{4} \rfloor$. The vertices $(w_{4i,1})$ are colored with the color c_5, where $1 \leq i \leq \lfloor \frac{m}{4} \rfloor$. The vertices $(w_{4i,2})$ are colored with the color c_6, where $1 \leq i \leq \lfloor \frac{m}{4} \rfloor$. The vertices $(w_{4i,3})$ are colored with the color c_8, where $1 \leq i \leq \lfloor \frac{m}{4} \rfloor$. The vertices $(w_{4i,4})$ are colored with the color c_{11}, where $1 \leq i \leq \lfloor \frac{m}{4} \rfloor$. The vertices $(w_{4i-2,j})$ are colored with the color c_{2j-2}, where $1 \leq i \leq \lfloor \frac{n+2}{4} \rfloor$, $5 \leq j \leq n$. The vertices $(w_{4i,j})$ are colored with the color c_{2j+3}, where $1 \leq i \leq \lfloor \frac{m}{4} \rfloor$ *and* $5 \leq j \leq n$.

Suppose $\chi_s(K_m[W_n]) > (2n+3)$. Then it contradicts, the chromatic number of star coloring. So $\chi_s(K_m[W_n]) \leq (2n+3)$. But it contradicts, the definition of star coloring that we need atleast 3 colors for any path on four vertices. So, $\chi_s(K_m[W_n]) = (2n+3)$, for m *and* $n \geq 4$.

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Sivakami Ramasamy
Department of Mathematics,
RVS College of Engineering and Technology,
Coimbatore-641 402,
Tamil Nadu, India
and,
Part-Time Research Scholar (Category-B),
Research & Development Centre,
Bharathiar University,
Coimbatore-641 046,
Tamil Nadu, India
email: *sivakawin@gmail.com*

Kaliraj Kalimuthu
Ramanujan Institute for Advanced Study in Mathematics,
University of Madras,
Chepauk, Chennai-600 005,
Tamil Nadu, India
email: *kalirajriasm@gmail.com*

Vernold Vivin Joseph
Department of Mathematics,
University College of Engineering Nagercoil,
Anna University Constituent College,
Nagercoil-629 004,
Tamil Nadu, India
email: *vernoldvivin@yahoo.in*